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Some Characteristics of Inelastic Proton-Nucleon Collisions Produced by Protons of Energy 6.2 GeV in Nuclear Emulsions (*).

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Summary. — Nuclear interactions produced by protons of energy 6.2 GeV in nuclear emulsions have been analysed to deduce information regarding some of the characteristics of proton-nucleon collisions. 703 disintegrations have been obtained by « along the track » scanning. Accurate measurements of multiple scattering and of grain density have been made on favourable tracks of relativistic secondary particles; using these measurements it has been possible to identify the particles up to the highest energies involved. It is found from this investigation that in collisions of protons of energy 6.2 GeV with nucleons: *a*) the mean charged pion multiplicity is 1.51 ± 0.18; *b*) the mean inelasticity is 0.43, and is, within errors, independent of the total multiplicity n_s; *c*) the nucleons recoiling in the C.M. system are very strongly collimated symmetrically in the forward and backward directions; *d*) the mesons created in the collision also show an appreciable amount of forward-backward collimation in the C.M. system.

1. – Introduction.

As a result of investigations carried out during the last five to ten years, it has been shown that some of the characteristics of nuclear interactions, (such as inelasticity and angular distribution of the secondary particles), in collisions

(*) The results were reported at the International Conference on High Energy Physics (of the IUPAP) held at Kiev in July 1959.

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induced by protons of energy less than about 2 GeV from accelerators are quite different from those observed in collisions of protons of energy grater than about 50 GeV from the Cosmic Radiation. It is, therefore, of interest to study these properties for the intermediate primary energy interval.

Until about the middle of 1958, when this problem was started, the majority of the investigations ([1-5]) carried out using nuclear emulsions, with protons of energy $(4 \div 10)$ GeV from the accelerators, were of a phenomenological nature. No serious attempt was made in these experiments to measure the energies of the shower particles by the method of multiple Coulomb scattering nor to identify them by the method of scattering and grain density.

Recently, KALBACH *et al.* ([6]) have made an analysis of proton-proton collisions using protons of energy 6.2 GeV from the Berkeley Bevatron. They have made measurements of angles, Coulomb scattering and grain density on tracks of shower particles and derived information regarding the properties of proton-proton collisions.

In this paper we report a similar experiment using emulsions exposed also to the 6.2 GeV direct proton beam of the Berkeley Bevatron. Our main object is to study the characteristics of proton-nucleon collisions at this energy. Because of the complex composition of the nuclear emulsion, it was necessary to adopt selection criteria to separate proton-nucleon interactions from those involving secondary collisions. In Appendix I, we have given reasons in support of our selection criteria. We have attempted to make accurate measurements of angles, Coulomb scattering and grain density on favourable tracks associated with the selected interactions.

The significant results obtained from this investigation concern the inelasticity of collisions at the incident proton energy of 6.2 GeV and the angular and energy distributions of the recoiling nucleons in the centre of mass system, hereafter called the C.M. system. It is found that the inelasticity of collisions has a large spread with a mean value of 0.43 ± 0.06; this has been obtained from two independent sets of measurements. Further, the observations show that there is very little or no dependence of inelasticity on the multiplicity of the star. The angular distribution of the nucleons in the C.M. system is very strongly peaked in the forward and backward directions.

([1]) W. R. JOHNSON: *Phys. Rev.*, **99**, 1049 (1955).

([2]) M. SCHEIN, D. M. HASKIN and R. G. GLASSER: *Nuovo Cimento*, **3**, 131 (1956).

([3]) U. HABERSCHAIM: *Nuovo Cimento*, **4**, 669 (1956).

([4]) N. P. BOGACHEV, S. A. BUNIATOV, U. P. MEREKOV and V. M. SIDOROV: *Dokl. Akad. Nauk SSSR*, **121**, 617 (1958).

([5]) N. P. BOGACHEV, VAN SHU-FEN, I. M. GRAMENITSKY, L. F. KIRILLOV, P. M. LEBEDEV, V. B. LIUBIMOV, P. K. MARKOV, M. I. PODGORETSKY, V. M. SIDOROV, K. D. TOLSTOV and M. G. SHAFRANOVA: *Atomnaya Energia*, **4**, 281 (1958).

([6]) R. M. KALBACH, J. J. LORD and C. H. TSAO: *Phys. Rev.*, **113**, 325 (1959).

2. – Experimental technique.

2·1. *Emulsions used.* – In this investigation we have made use of two single G-5 emulsions 600 μm thick and of size 18 cm×12 cm exposed to the 6.2 GeV direct proton beam of the Berkeley Bevatron (*). The mean angle made by the beam protons with respect to the plane of the emulsion is about 0.5°. These emulsions originally formed part of a big stack used by the Paris group (7) for work on K-mesons.

2·2. *Method of scanning.* – For this experiment it is important to have a large number of proton-nucleon interactions selected without any kind of systematic bias, particularly regarding the size of the star, in order that the mean values of quantities relating to nucleon-nucleon collisions, derived from these observations may have a real meaning. When a primary proton makes a collision with a free proton, it usually results in a star with no or one associated heavy prong. On the other hand, if it makes a collision with a peripheral nucleon of a nucleus of the emulsion, without further secondary collisions which would appreciably alter the angles and energies of the particles emerging from the first collision, then also one would expect that these interactions will have no or very few heavy prongs.

To obtain such interactions, we adopted the « along the track » scanning method. Tracks due to the beam particles and lying in the central regions of the emulsion, (between about 50 μm from either of the emulsion surfaces) were followed until they produced an interaction or left this useful thickness of the emulsion. This procedure ensured that all interactions were located in regions of the emulsion favourable for various observations to be made on them.

In the recent work by KALBACH *et al.* (6), they have followed flat tracks of secondary relativistic particles back to their origins. Two important consequences of this will be: i) a systematic preference for stars of higher multiplicity and, ii) necessity of large corrections for the number of tracks at very small angles and very large angles.

2·3. *Interactions chosen for analysis.* – Using the procedures described above, we obtained a total of 703 stars. In this we have included large angle deviations greater than 5°. We have excluded from this selection all events which

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(*) We are grateful to Professor LEPRINCE-RINGUET of the Ecole Polytechnique, Paris, for kindly lending these to us.

(7) J. CRUSSARD, V. FOUCHÉ, J. HENNESSY, G. KAYAS, L. LEPRINCE-RINGUET, D. MORELLET and F. RENARD: *Nuovo Cimento*, **3**, 731 (1956).

are consistent with elastic proton-proton collisions; the procedure adopted for this is described in Appendix I. Out of these 703 interactions, 212 were obtained from an initial scan of a region of the plate where the track density was very high. It was found that the efficiency for detecting stars with $N_h = 0$ or 1 in this region of the emulsion was only about 50% compared to that for regions where the track density was relatively very low and where the remaining 491 interactions were found.

All the ensuing analysis, *unless otherwise mentioned*, is based on the 491 interactions which also include 100 reported in an earlier investigation ([8]). Arguments are given in Section **3**, to show that the efficiency for detecting stars with no or one heavy prong, (among the 491 interactions), should be almost 100%.

2·4. *Nomenclature.* – In this paper we employ, with the following modifications, the nomenclature introduced by the Bristol group ([9,10]) for describing the size of the stars.

In the emulsions employed, the 6.2 GeV proton tracks had a grain density of 22.8 per 100 μm. Tracks with grain density less than 39.8 per 100 μm (corresponding to 1.55 times g/g_{plateau}) and also all pion tracks of grain density $\geqslant 39.8$ per 100 μm are classified as shower particles. Tracks with grain densities between 39.8 and 126 per 100 μm but excluding pion tracks, are classified as « greys » and those with grain densities greater than 126 as « blacks ». (Black tracks do not include recoils of range < 5 μm). For protons this classification corresponds to energies greater than 300 MeV for « shower tracks », between 300 and 25 MeV for « grey tracks » and less than 25 MeV for « black tracks ». If N_b, N_g and n_s are the numbers of black, grey and shower particle tracks associated with a star, the star can be represented as $N_b + N_g + n_s$. When the sum $N_b + N_g$ is written as N_h, the number of heavy prongs, the star is represented as $N_h + n_s$.

3. – Selection of proton-nucleon collisions.

The best method of selecting nuclear interactions which are in the main due to proton-nucleon collisions would be to include only stars of the type $0+0+n_s$ and $0+1+n_s$ in which the grey proton is emitted, in the laboratory

([8]) M. V. K. Appa Rao, R. R. Daniel and K. A. Neelakantan: *Proc. Ind. Acad. Sci.*, **43** A, 181 (1956).

([9]) R. H. Brown, U. Camerini, P. H. Fowler, H. Heitler, D. T. King and C. F. Powell: *Phil. Mag.*, **40**, 862 (1949).

([10]) U. Camerini, J. H. Davies, P. H. Fowler, C. Franzinetti, W. O. Lock, H. Muirhead, D. H. Perkins and G. Yekutieli: *Phil. Mag.*, **42**, 1241 (1951).

(L)-system, in the forward direction with respect to the incoming particle. These will be due either to collisions made by the primary protons with hydrogen nuclei—« free protons »—in the emulsion or with neutrons or protons in the peripheral regions of heavier emulsion nuclei. These events constitute only about 15% of the total number of interactions.

In the case of stars with larger number of heavy prongs, it is natural to expect that secondary collisions in the same target nucleus have taken place. A detailed study has been made to determine the maximum value of N_h that can be considered such that one still gets a fairly rich sample of proton-nucleon collisions without their characteristics getting appreciably altered by secondary collisions. The details of this analysis are given in Appendix I. It is shown that the basic characteristics of proton-nucleon collisions, such as the multiplicity and angular distribution of shower particles, remain reasonably unaltered for values of $N_h \leqslant 5$. It is found that for the same value of N_h, these characteristics depend on N_g. Therefore, for our final analysis we have used only events with $N_h \leqslant 5$ and $N_g = 0$; cases with $N_g = 1$ are also included if the grey track lies in the forward hemisphere. These we will designate as *class* A stars.

Though the above selection has been shown to be rich in proton-nucleon collisions, for obtaining more precise information regarding quantities such as the energy and angle of « grey » protons in the C.M. system—these are the protons which recoil backwards in the C.M. system—or the multiplicity in the elementary proton-nucleon collisions, it is necessary to use only pure events of the type $0+0+n_s$ and $0+1+n_s$ in which the grey proton is emitted in the forward direction. This group of stars is referred to as *class* B. We will give arguments at the end of this section to show that this group consists almost exclusively of proton-nucleon interactions.

We have a total of 196 interactions belonging to class A. Among these 72 belong to class B. The 72 interactions of class B consist of 33 of the type $0+1+n_s$ and 39 of the type $0+0+n_s$. (Three examples of type $0+1+n_s$, in which the grey track was in the backward direction have been excluded from class B.)

It may be argued that some of the inelastic proton-nucleon collisions may result in stars of the type $1+0+n_s$. In our analysis there are 25 such examples selected as due to inelastic collisions. Among these there are 11 in which the black track is in the forward hemisphere and 14 in which it is in the backward hemisphere. This very strongly suggests that almost all of these black tracks arise in evaporation processes. Further, it can be seen from considerations of energy and of angle of emission of the recoiling protons in the C.M. system, that not many can give rise to black tracks in the L-system. We therefore, have not included stars of the type $1+0+n_s$ in class B.

Stars of this group with an even total prong number, (N_g+n_s), result from

collisions made by the incoming proton with a free or a bound proton in the emulsion, whereas those with odd number of prongs result only from collisions made with bound neutrons. Among the 72 examples belonging to class B, there are 48 with even number of prongs and 24 with odd number of prongs. If we assume that the number of protons is equal to the number of neutrons on the surface of nuclei, we can say that 24 of the stars with even number of prongs should be due to proton collisions with bound protons and the remaining 24 due to proton collisions with free protons in the emulsion. Using the composition of the emulsion as supplied by the manufacturers and assuming that the mean free path for nuclear interactions of 6.2 GeV protons in emulsions is 35 cm ([2,8]), we estimate that the 24 collisions involving free protons correspond to a cross-section for inelastic proton-proton collision of (41 ± 9) mb. This value is not inconsistent with the known value of about 30 mb at these energies ([5,11,12]). Furthermore, it may be seen from Fig. 6 that there are five events of the type $0+1+1$ in which the grey particle has an energy in the C.M. system corresponding to γ_c of the system. Since these could be elastic collisions, we have re-examined the events and found that if one is prepared to stretch the limits of error in the measurements by a factor of two, then two of these five events can be attributed to elastic collisions. If this be so then we would get, for inelastic proton-proton collisions, a cross-section of 38 mb. From this, we infer that our scanning efficiency for stars belonging to class B is nearly 100%.

This can also be seen from the following. Corresponding to the 48 examples of inelastic proton-proton collisions (both with free and bound protons) estimated earlier in this section, we found 14 examples of elastic proton-proton collisions (see Appendix I). It is known that at these energies the cross-section for elastic proton-proton collision is about 8 mb ([13,6]) as compared to 30 mb. for inelastic collisions. From this, one clearly sees that the number of inelastic proton-proton collisions we have obtained, 48, is about the right number to be expected.

4. – The proportion of protons among shower particles.

The secondary shower particles associated with these stars consist of pions, protons recoiling in the forward hemisphere of the C.M. system (hereafter called « forward » protons), and a small proportion of protons recoiling in the

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([11]) W. B. Fowler, R. P. Shutt, A. M. Thorndike, W. L. Whittemore, V. T. Cocconi, E. Hart, M. M. Block, E. M. Harth, E. C. Fowler, J. D. Garrison and T. W. Morris: *Phys. Rev.*, **103**, 1489 (1956).

([12]) R. Cester, T. F. Hoang and A. Kernan: *Phys. Rev.*, **103**, 1443 (1956).

([13]) B. Cork, W. A. Wenzel and C. W. Causey, jr.: *Phys. Rev.*, **107**, 859 (1957).

backward hemisphere of the C.M. system (hereafter called « backward » protons). The procedure adopted to identify individual particles is described in Appendix III. The final results obtained are summarized in Tables I and II.

TABLE I. – *Proportion of protons and pions among the shower particles as a function of* $p\beta$.

$p\beta$ MeV/c	Forward proton		Backward proton		Charged pions	
	class A	class B	class A	class B	class A	class B
< 1500	0	0	83.6 (16)	18.6 (5)	197.3 (32)	66.4 (12)
1500—2500	30.5 (11)	8.0 (3)	11.6 (1)	0	14.5 (5)	1.0 (1)
> 2500	87.9 (42)	24.3 (12)	6.4 (1)	0	14.1 (6)	6.8 (2)

The figures in brackets are the observed number of tracks.

TABLE II. – *Proportion of protons and pions among the shower particles as a function of the angle of emission in the* L*-system.*

Angle θ	Forward proton		Backward proton		Charged pions	
	class A	class B	class A	class B	class A	class B
0—10°	105.0 (50)	32.3 (15)	18.8 (6)	7.8 (2)	55.2 (20)	21.9 (7)
10°—30°	12.8 (3)	0	49.7 (9)	10.8 (3)	56.6 (14)	26.1 (6)
$> 30°$	0	0	33.1 (3)	0	114.1 (9)	26.2 (2)

The figures in brackets are the observed number of tracks.

Class A stars included in Tables I and II are from the total of 703 stars while class B stars are from the efficient scan only. The numbers given in these tables were obtained after correcting the observed number of tracks for geometric loss, which depends on the angle of emission.

The following observations can be made from these Tables:

i) The relative numbers obtained for class A and class B stars in Tables I and II are in fair agreement, which indicates that the vast majority of class A stars have characteristics consistent with a single proton-nucleon collision.

ii) About 25% of the shower particles with $p\beta < 1500$ MeV/c are protons, the remaining being pions, while for $p\beta > 2500$ MeV/c about 85% are protons and the remaining pions.

iii) Almost all the « forward » protons are contained within an angle of 10° with respect to the forward direction of the primary in the L-system. The three particles in class A observed in the interval $10° \div 30°$ (with angles 14.8°, 15.8° and 18.4°) give an indication of the secondary collisions suffered by the forward protons. It is found that about 70% of the shower particles within an angle of 10° are protons and the rest pions, whereas at angles $> 10°$, only about 35% are protons and the remaining pions.

5. – The number of « forward » and « backward » protons per star.

5·1. « *Forward* » *protons* $\langle N_{fp} \rangle$. – The average number $\langle N_{fp} \rangle$ of « forward » protons per star can be calculated from the number of such identified protons in class A and class B stars separately. We have done this in two ways:

i) Using the length criterion adopted for selecting tracks for measurement, we can attribute to each identified particle a geometric correction factor which will depend on the angle of emission. Thus one can directly estimate the total number of such particles from the number of identified ones.

ii) One can also calculate $\langle N_{fp} \rangle$ from the ratio of identified forward protons to all identified shower particles within an angle of 10°, and the total number of shower particles also within the same angular interval.

The two methods are not completely independent. The results obtained are summarized in Table III.

TABLE III. – *Average number of forward protons per star.*

Type of star	Number of forward protons per star $\langle N_{fp} \rangle$	
	Method 1	Method 2
Class A	0.42 ± 0.07	0.42 ± 0.07
Class B	0.39 ± 0.11	0.47 ± 0.13

It can be seen from this table that, within experimental errors, the values of $\langle N_{fp} \rangle$ are all consistent with 0.5. From cosmic ray experiments it has been deduced by VOROBIEV [14] that at primary proton energies of $(3 \div 10)$ GeV, the proton has a tendency to persist and that $\langle N_{fp} \rangle$ has a value between 0.7 and 0.8. From our observations it seems unlikely that $\langle N_{fp} \rangle$ could be greater than 0.6.

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[14] V. A. VOROBIEV: *Žurn. Èksp. Teor. Fiz.*, **33**, 264 (1957).

5·2. *»Backward» protons* $\langle N_{bp} \rangle$. – From class B stars the mean number of « backward » protons $\langle N_{bp} \rangle$ per star was calculated to be 0.62 ± 0.10 after estimating the total number of « backward » protons in the manner described in Section **7**.

6. – Multiplicity.

For determining the mean multiplicity $\langle n_s \rangle$, it is important to ensure that shower particles are detected with a high degree of efficiency. The efficiency in this investigation is very nearly 100% and can be demonstrated in the following manner.

i) The sizes of 50 stars of all types having a total of 106 shower particles were independently determined by a second observer. In this determination, one additional shower particle track was found and one missed.

ii) In the preceding section it was shown that there was a significant excess of even pronged stars in class B and this was satisfactorily attributed to the expected contribution from collisions with free protons. If there was a significant loss of shower particles in determining the star sizes, the even-odd effect would have been considerably reduced.

In Fig. 7, we have plotted the mean multiplicity $\langle n_s \rangle$ as a function of N_h. It shows very clearly that, at this energy, secondary collisions in the target nuclei result in the production of a larger number of heavy prongs and of shower particles. It is found that among stars of class A the mean multiplicity is reasonably constant for values of N_h up to 5 and has a mean value $\langle n_s \rangle_A = 2.28 \pm 0.16$. For stars of class B, $\langle n_s \rangle_B = 2.11 \pm 0.25$; in this class the mean multiplicities for the 48 proton-proton interactions (stars of even prong number) and the 24 proton-neutron interactions (stars of odd prong number) are 2.18 and 1.96 respectively.

The shower particles contain pions and protons. (At this energy the cross-section for production of K-mesons is negligible compared to that for production of pions and therefore the presence of K-mesons and hyperons has been ignored.) In Section **4**, we have shown how the fraction of protons among the shower particles was determined. After making this correction for the shower particles associated with class B stars, we obtain for the mean charged pion multiplicity $\langle n_{\pi^\pm} \rangle_B$ a value of 1.51 ± 0.18. This may be compared with the value of 1.9 ± 0.3 obtained by KALBACH *et al.* [6] for proton-proton collisions at the same primary energy. The corresponding value for class A stars is 1.68 ± 0.12. Assuming the ratio $N_{\pi^\pm}/N_{\pi^0}$ to be 2, we obtain for class A and class B stars total pion multiplicities of 2.52 ± 0.18 and 2.27 ± 0.27 respec-

tively as compared to the value of 2.8 obtained by KALBACH *et al.* Within experimental errors these values agree; but as pointed out in Section 2, the scanning method used by these authors favours to some extent stars of high multiplicity. This may be a possible reason why their cross-sections for such stars are significantly larger than that predicted by statistical theories ([15,16]). The mean multiplicity (for all pions) obtained from Fermi's ([15]) theory is 2.5. BARAŠENKOV *et al.* ([17]) give for this a value 2.9; this is based on an intermediate coupling assumption (assumption W_2); the strong coupling assumption (assumption W_1) of these authors yields a value of 2.7 but has also the consequence that the cross-section for K-meson production is much higher than experimentally observed. These values may be compared with our experimental value for multiplicity of 2.27.

7. – Energy determination of the primary proton using the angular distribution of the shower particles.

It is of interest to investigate whether a true estimate of the primary energy can be obtained by using the angular distribution of the shower particles. In most of the investigations of high energy nuclear interactions it is customary to estimate the energy of the primary particle from the angular distribution of the shower particles. In this method it is assumed that: i) the shower particles are emitted symmetrically in the forward and backward directions in the C.M. system and ii), the ratio $\overline{\beta}_p/\overline{\beta}_s$ of the velocity $\overline{\beta}_p$ of the colliding nucleons to that, $\overline{\beta}_s$, of the secondary particles in the C.M. system is ≈ 1. One can then measure the median angle θ_M which contains half the number of shower particles and estimate the primary energy. In investigations with mono-energetic protons from accelerating machines, one can obtain a combined angular distribution for shower particles associated with a large number of interactions (a composite star) and determine θ_M accurately.

An attempt along these lines was made by HABERSCHAIM ([3]) using disintegrations produced by 6.2 GeV protons. The primary energy for all stars with $N_h \leqslant 7$, obtained by this method was (11.7 ± 2.2) GeV. It was pointed out that the energy derived thus, contrary to common belief, was more likely to be too large than too small. It was further suggested in that paper that this effect might be partly due to the protons which may not satisfy the velocity conditions in the C.M. system.

([15]) E. FERMI: *Prog. Theor. Phys.*, **5**, 570 (1950).

([16]) J. V. LEPORE and R. N. STUART: *Phys. Rev.*, **94**, 1724 (1954).

([17]) V. S. BARAŠENKOV, B. M. BARBAŠEV and E. G. BUBELEV: *Suppl. Nuovo Cimento*, **7**, 117 (1958).

As will be shown in Section **8**, the protons recoiling backward in the C.M. system will emerge in the vast majority of cases as grey particles, while the « forward » protons will always be found as shower particles at small angles ($< 10°$) in the Laboratory system. Further, a small proportion of protons emitted backward in the C.M. system with low energies will also be contained as shower particles within angles $< 25°$ in the L-system. In Section **4**, we have deduced quantitative values for the proportion of protons among the shower particles as a function of the space angle θ. Using these correction values, we estimated θ_M for the charged pions in class A stars and obtained a value of 28°. This corresponds to a primary energy of (4.8 ± 1.2) GeV. (Owing to poor statistics the energy obtained in this manner for class B stars has a very large error; the value is (9.0 ± 3.0) GeV.) We would like to mention that if no correction was made for the protons among the shower particles the energy estimates turn out to be 15 GeV and 25 GeV for class A and class B stars, respectively.

Thus one may conclude from this that the median angle method can be usefully employed to determine low primary energies, (between 5 and 50 GeV) if one can correct for the protons among the shower particles, as also to determine very high primary energies where the meson multiplicity is very large.

8. – Energy and angular distribution of protons and pions in the C.M. system.

8·1. *Protons*:

i) « Forward » protons. – As has been shown earlier, almost all these protons are emitted at angles $< 10°$ in the L-system. Further, they should have energies > 1.0 GeV (corresponding to γ_c of the C.M. system) in the L-system. At these energies, any secondary collision suffered by these protons, involving only small energy transfers, will not seriously affect their energies and angles of emission. We, therefore, have used in this analysis class A stars from our entire scanning data.

Using the procedures described in Appendix III, we identified 53 particles as protons recoiling forwards in the C.M. system, of which only 11 had dip angles δ, such that $\operatorname{ctg} \delta \geqslant 100$. It is shown in Appendix III that there is an increase of spurious scattering for tracks with $\operatorname{ctg} \delta < 100$ and the energy estimates therefore tend to be low in the region of $p\beta > 2.5$ GeV/c; however, the measurements still allow particle identification. Since the angles which these particles make in the C.M. system are not very sensitive to a little underestimation of their energies in the L-system, we have calculated for particles with $\operatorname{ctg} \delta > 30$ their angle; in the C.M. system using the usual Lorentz trans-

formation. For a given selection criterion, the useful solid angle decreases with increasing angle of emission. We therefore gave each particle a geometric correction factor to compensate for this effect. The angular distribution in the C.M. system is shown in Fig. 1*a*. In order to illustrate the effect of increased spurious scattering for smaller values of ctg δ, we have also included in this figure the distribution obtained from all measured tracks.

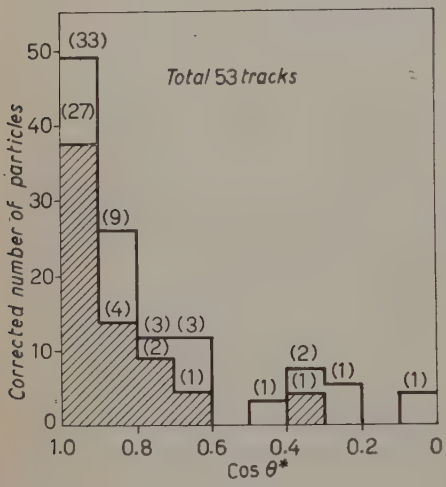

Fig. 1*a*. – Angular distribution of « forward » protons (corrected for geometry) in the C.M. system for class A stars. The tracks with ctg $\delta \geq 30$ are hatched. The observed numbers are given inside brackets.

ii) « Backward » protons. – In proton-nucleon collisions at this energy, the protons recoiling backwards in the C.M. system will usually emerge in the L-system with energies < 1.0 GeV and at angles large compared to the « forward » protons. It is obvious that if these protons suffer even secondary elastic collisions in the same target nucleus, their energies and angles could alter appreciably, resulting in significant differences in their C.M. characteristics derived from such measurements. Hence, in this analysis we have used only class B stars from the efficient scanning.

Among the backward recoiling protons, those with energy < 400 MeV in the L-system will have grain densities > 33 per 100 μm and, therefore, their energies can be obtained from measurements of grain counting alone. The contamination of pions in this grain density interval (energy < 60 MeV) is known to be very small. Even this was corrected by making rough estimates of the scattering in all tracks with grain density > 33 per 100 μm (including those not selected for regular scattering measurements). We found 3 tracks in a total of 154, and identified them as due to pions by this method. These were also included in n_s according to our nomenclature in Section 2. A total of 39 such events with a backward recoiling proton of energy < 400 MeV were obtained from the 72 class B stars. There is no loss factor in the selection of events with associated protons of energy < 400 MeV. The energies of individual grey particles were estimated from measurements of grain density on the assumption that they are protons. The energy-grain density calibration curve was obtained from measurements on tracks of flat stopping π^+ mesons.

As regards « backward » protons recoiling with energies > 400 MeV in the L-system, one sees from Fig. 9 that those whose tracks are favourable for measurements of $p\beta$ and grain density can be identified when the Lorentz transformations are made for energy and angle. We had only two « backward » protons so identified amongst a total of 15 tracks on which these measurements were made. From geometrical considerations we estimate that there should have been five other «backward» protons in this energy interval which were missed because of our selection criteria. We ascribe to these five particles angles and energies similar to the two identified protons. (It can be seen from Figs. 1 and 2 that the number of « backward » recoiling protons with energies > 1 GeV in the L-system should be negligible.) The angles and energies of these protons in the C.M. system were calculated using the Lorentz transformation relations. The results are represented in Figs. 1*b*, 2.

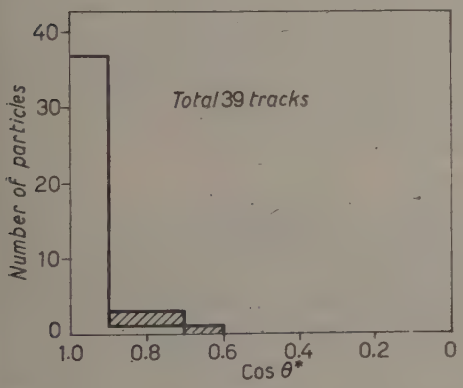

Fig. 1*b*. – Angular distribution of « backward » protons in the C.M. system for class B stars. The hatched area represents the correction for the loss of particles with energies >400 MeV in the L-system.

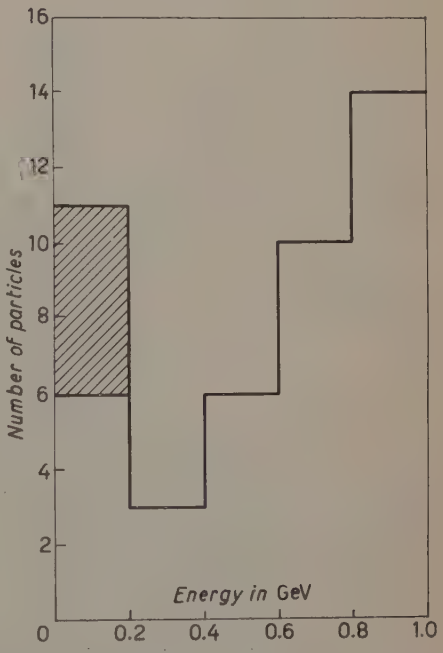

Fig. 2. – Energy distribution of « backward » protons in the C.M. system for class B stars. The hatched area corresponds to the loss of particles with energies > 400 MeV in the L-system.

We have also studied the distribution of the transverse momentum for the backward protons. The results are shown in Fig. 3*a*. It is found that it has a maximum at about 200 MeV/c and the mean value is (338 ± 56) MeV/c.

iii) Results. – In Figs. 1*a*, 1*b* and 2, are plotted the energy and angular distributions for the « forward » and « backward » recoiling protons in the C.M. system. Measurements made on the « backward » protons are more accu-

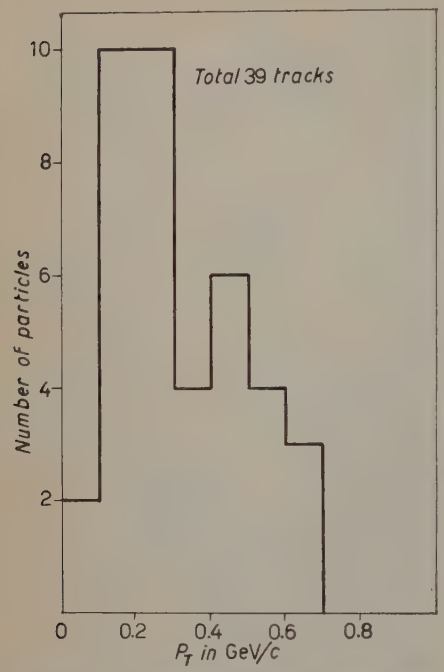

Fig. 3*a*. – Transverse momentum distribution of « backward » protons for class B stars.

rate and less subject to systematic errors. It is seen from Figs. 1*a*, 1*b* that the recoiling protons in the C.M. system have a striking collimation about the line of motion of the incident proton in the forward and backward directions in a symmetric manner, the median angle being 14.5° for the backward protons. KALBACH *et al.* (6) have obtained a similar angular distribution; but the tail of their distribution extends right up to 90° as in the case of our distribution for forward protons of all ctg δ values (Fig. 1*a*). Their angular distribution was also obtained from measurements of $p\beta$ and grain density; but they have not mentioned their criteria for selection of tracks.

The energy distribution of the « backward » protons has a large spread and the mean value is found to be (596 ± 90) MeV. If one uses for the « forward » protons this energy distribution and the angular distribution given in Figs. 1*a*, 1*b* both in the C.M. system, then in the L-system they all emerge within 10° with respect to the forward direction of the incident proton. This can be demonstrated directly in the following manner. In Fig. 4 we have plotted the number of shower particles per steradian as a function of angle in the L-system for class B stars. It seems reasonable from this figure that the peak at small angles should be due to the « forward » protons

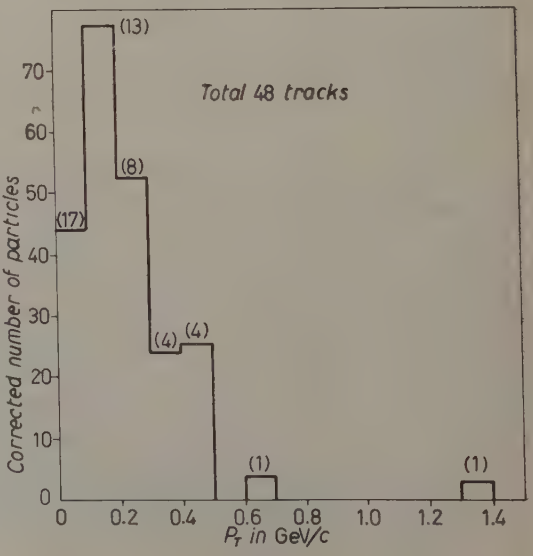

Fig. 3*b*. – Transverse momentum distribution of pions (corrected for geometry) for class A stars. The observed numbers are given inside the brackets.

whereas the distribution for angles $> 10°$ should be due to pions and « backward » protons. We have extrapolated this part of the curve up to 0° and estimated the number of « forward » protons within the angular interval $(0 \div 10)°$ and get a value of 44 as compared to 25 pions and « backward » protons in the same angular interval. These numbers are consistent with the corresponding 32.2 (15) and 29.7 (9) obtained directly from scattering and grain density measurements (Table II). The numbers within brackets are the observed numbers.

From the above observations one can conclude that there is complete symmetry between the forward and backward recoiling protons in the C.M. system regarding the energy and angular distributions.

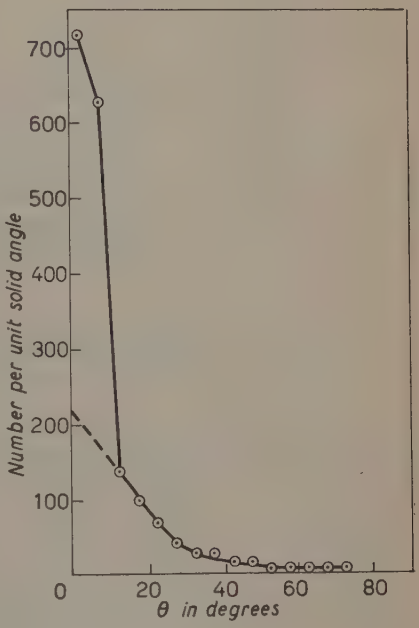

Fig. 4. – Number of shower particles per unit solid angle in the L-system as a function of space angle for class A stars. The dashed line is the extrapolation for the distribution of pions and « backward » protons.

8·2. *Pions*:

For the study of the characteristics of pions in the C.M. system we have used stars of class A from the entire scanning data. In order to be selected for measurements, tracks of particles emitted at angles $\leqslant 10°$ and $> 10°$ should have lengths per plate $\geqslant 6$ mm and $\geqslant 3$ mm respectively. We obtained a total of 43 pion tracks using the identification procedure described in Appendix III. In addition we obtained five low energy pions which were identified from a visual estimate of the small angle scatterings present in the tracks produced by them. Correction factors have been assigned to the 43 tracks to compensate for the geometric loss. Using the measured energies and angles, the corresponding quantities in the C.M. system were calculated for each pion. The results are shown in Figs. 5*a*, 5*b*.

The angular distribution of pions in the C.M. system shows that there is to some extent forward-backward collimation in their direction of emission with respect to the primary proton and within experimental errors there is symmetry. This is in contradiction to the isotropic distribution for neutral

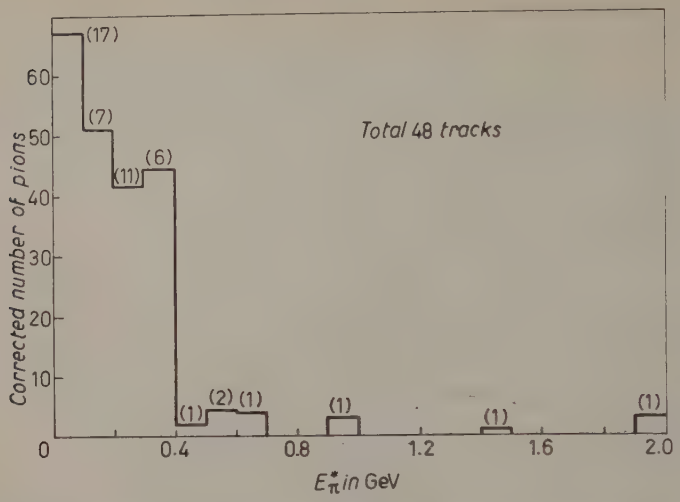

Fig. 5*a*. – Energy distribution of pions in the C.M. system for class A stars. The observed numbers are given inside brackets.

pions obtained by KING ([18]). The energy distribution of pions has a large spread and the mean energy is (290 ± 45) MeV compared to the value of (236 ± 20) MeV obtained by KALBACH *et al* ([6]) for a total of 143 pions.

We have also studied the distribution of the pion transverse momentum. The results are shown in Fig. 3*b*. There is a maximum at about 200 MeV/c and the mean value is (225 ± 33) MeV/c.

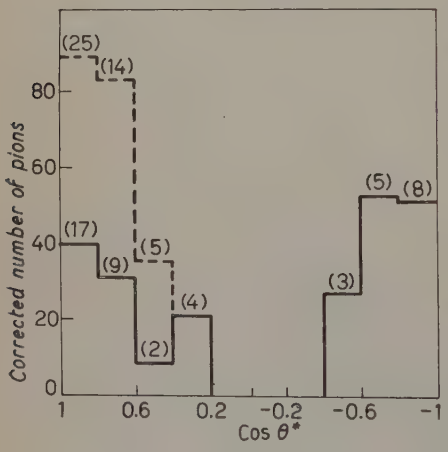

Fig. 5*b*. – Angular distribution of pions in the C.M. system. The dashed histogram corresponds to the folded distribution.

9. – The inelasticity of collisions.

The inelasticity k of nucleon-nucleon collision at high energies is defined as the ratio of the total energy radiated as pions and other particles created in the collision, to the total available energy in the C.M. system. This para-

([18]) D. T. KING: *Phys. Rev.*, **109**, 1344 (1958).

meter has become of great importance for our understanding of very high energy collisions particularly in the study of Extensive Air Showers. Some attempts have also been made by other workers ([5,6]) to estimate k at machine energies. In this section we describe two independent methods by which we estimate k.

9·1. *From the mean multiplicity of pions and their mean energy in the* C.M. *system.* – We have shown in Section **6**, that the mean number of pions $\langle n_{\pi^0} \rangle_B$ 2.27 ± 0.27. In Section **8**, we showed, from measurements made on 48 charged pions, that the mean kinetic energy in the C.M. system is 290 MeV as compared to the value of 236 MeV obtained by KALBACH *et al.* from 143 tracks. Therefore, using a weighted mean value of 250 MeV for the mean kinetic energy of pions in the C.M. system we calculate:

$$k = \frac{2.27(250 + 138)}{2\,020} = 0.44 \pm 0.03 .$$

On the other hand if we use only our results we get a value $k = 0.48 \pm 0.07$. KALBACH *et al.* ([6]) have also calculated k using this method and they get a value of 0.49 ± 0.05.

9·2. *From the energy distribution of protons recoiling backward in the* C.M. *system.* – In the preceding section we showed that it is possible to identify without any systematic bias, almost all events from class B stars in which there is an associated proton recoiling backwards in the C.M. system. Since we also showed in Section **8**, that the forward and backward recoiling protons are symmetric in the C.M. system, we can assume that, on the average, the forward recoiling protons carry away the same amount of energy in the C.M. system as the backward recoiling protons. The mean proton energy in the C.M. system obtained in Section **8**, is 596 MeV. From this we get for the inelasticity a value $k = 0.41 \pm 0.09$.

Thus it is found that the values of inelasticity estimated by two independent methods are consistent with $k = 0.43$. An attempt was also made to study whether k depends on n_s. Class B stars were divided into two groups, one with $n_s \leqslant 2$ and the other with $n_s > 2$ and k was calculated for these two groups and found to be equal to 0.39 (from 27 events) and 0.44 (from 12 events) respectively. Since the errors on the individual values are large, no definite conclusion can be drawn from this. We have tried to deduce further information regarding this in the following manner. In Fig. 6 is plotted, for class B stars, the energy of the backward protons in the C.M. system for each event, against the multiplicity n_s associated with that event. The error in the measurements of grain density and angle of the grey tracks is such that the corres-

ponding error in the calculated energy in the C.M. system is about 5% and never greater than 10%. Therefore, the spread in the energy values is genuine

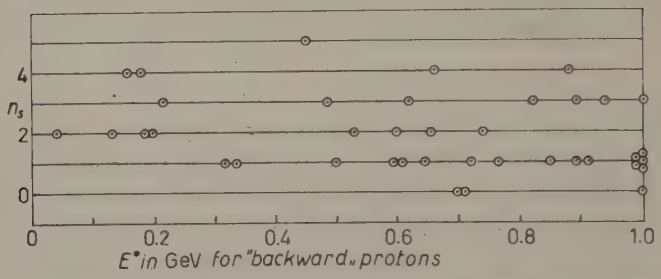

Fig. 6. – Plot of energy of the « backward » proton in the C.M. system against the multiplicity n_s for individual events of class B stars.

and not due to experimental errors. No definite dependence of k on n_s appears from this. The Bristol group ([19]) have found no dependence of k on n_s for primary energies $> 10^{11}$ eV.

10. – Summary.

i) In order to study the characteristics of nucleon-nucleon collisions at very high energies (jets) in nuclear emulsions, it is usual to select stars with N_h values up to a certain maximum. This maximum value of N_h varies from investigation to investigation and is decided to some extent in an arbitrary fashion. (Usually a compromise is made between the number of events available and the maximum value of N_h.) In this analysis we have tried to estimate this maximum value of N_h from a systematic study of the characteristics of the stars. It is shown that at 6.2 GeV, stars with $N_h \leqslant 5$ but $N_g = 0$ or 1 and with the grey track in the forward direction represent a fairly rich selection of proton-nucleon collisions. Of course, if one wants to make a richer selection than this, one will have to use events of the type $0+0+n_s$ and $0+1+n_s$ in which the grey track is in the forward direction.

ii) It is shown that by making accurate measurements of grain density and multiple scattering on favourable tracks, it is possible to identify protons and pions with reasonable confidence even in the region of $p\beta$ between 2.5 and 7 GeV/c. Our measurements suggest that the relativistic increase of grain

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([19]) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds: *Phil. Mag.*, **3**, 237 (1958).

density in the region of total energy $\gamma = 3 \div 10$ is smaller than that obtained by the Bristol group ([19,20]); the total relativistic increase, however, is about 12%.

iii) The mean number of protons recoiling in the forward emisphere in the C.M. system is, within experimental errors, 0.5. There is no indication at this primary energy that this value is significantly greater than 0.5 as claimed by VOROBIEV ([14]) from observations made on cosmic ray stars of energy between 3 and 10 GeV.

iv) The mean number of shower particles associated with proton-nucleon interactions is 2.1 ± 0.3 while the mean number of pions (charged and neutral) is 2.3 ± 0.3. The latter value may be compared with that of 2.5 predicted from Fermi's theory ([15]) and of 2.9 predicted by BARAŠENKOV *et al.* ([17]) (assumption W_2).

v) It is found that the median angle method can be usefully employed to estimate the primary energy, if proper precautions are taken to correct for the protons among the shower particles.

vi) The angular distribution of the recoiling protons in the C.M. system is remarkably collimated in the forward and backward directions and is symmetric, the median angle being about 15°. Further, the mean transverse momentum is 340 MeV/c. The « forward » nucleon emerges in the L-system at very small angles with respect to the direction of the incident proton. It seems likely that in collisions involving primary energies > 10 GeV, one of the nucleons will emerge in the L-system without any appreciable change in direction and with high energy whereas the other nucleon will carry away very little energy.

vii) The angular distribution of pions also shows an appreciable collimation in the C.M. system. The mean energy is found to be (290 ± 45) MeV and the mean transverse momentum to be 225 MeV/c.

viii) The inelasticity of proton-nucleon collisions (in the C.M. system) is 0.43 ± 0.06. This value is obtained independently from: *a*) the mean energy and mean multiplicity of pions in the C.M. system and, *b*) the mean energy of the recoiling protons in the C.M. system. There is no definite indication of a dependence of multiplicity on inelasticity. Whilst the mean value of inelasticity is 0.43 the actual values range from 0.1 to 0.9.

* * *

We wish to express our thanks to Professor M. G. K. MENON for useful discussions. We also wish to thank Mrs. T. M. UPADHYAY for the scanning of the emulsions.

([20]) R. R. DANIEL, J. H. DAVIES, J. H. MULVEY and D. H. PERKINS: *Phil. Mag.*, **43**, 753 (1952).

Appendix I

Selection of proton-nucleon collisions.

a) *Elastic collisions.* – Events were classified as elastic proton-proton collisions if they satisfied the following conditions:

i) The star is of the type 1+0+1 or 0+1+1 and the three prongs, including the primary, are coplanar within experimental errors. The heavy track should, wherever favourable, be consistent with having been produced by a proton.

ii) The dynamics of the events are consistent with elastic proton-proton collisions.

14 elastic proton-proton collisions satisfying these conditions were found. Large angle deviations less than 5°, were attributed to elastic proton-neutron collisions.

b) *Selection of inelastic proton-nucleon collisions.* – In nuclear disintegrations two measurable quantities which can give us information regarding secondary collisions in the same target nucleus are the size of the star and the angular distribution of the shower particles. For studying these characteristics, we have made use of the total number of stars obtained, (703), because the efficiency of detection of stars does not seriously depend on the multiplicity of shower particles nor their angular distribution.

i) The size of the star: In Fig. 7, we have plotted the mean value $\langle n_s \rangle$ corresponding to different values of N_h. It is clear from this figure that the multiplicity increases significantly with N_h and is presumably due to an increased number of secondary collisions. In the same figure we have also plotted the mean multiplicities for class A stars as a function of N_h. (Class A stars are those with $N_h \leqslant 5$, but $N_g = 0$ or 1 in the forward direction.) From this we infer that the mean multiplicity remains constant within the experimental errors for all values of N_h in class A. The mean multiplicities for class A stars and class B stars are 2.28 ± 0.16 and 2.11 ± 0.25, respectively. (Class B stars are those with $N_h = 0$, but $N_g = 0$ or 1 in the forward direction.)

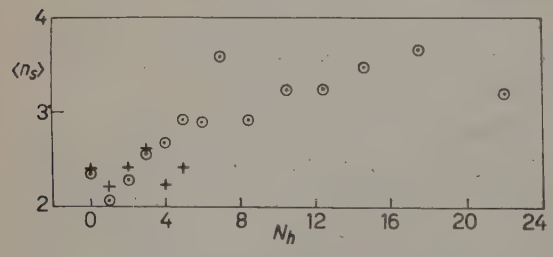

Fig. 7. – Mean multiplicity $\langle n_s \rangle$ as a function of N_h. The circles and crosses denote all stars and class A stars respectively.

ii) The angular distribution of shower particles: In Fig. 8, we have plotted the median angle of shower particles as a function of N_h for

all stars. In the same figure we have also plotted the values for class A stars. There is indication that for class A stars the median angle remains reasonably constant. For class A and class B stars the median angles are $20° \pm 1.5°$ and $15° \pm 2°$, respectively.

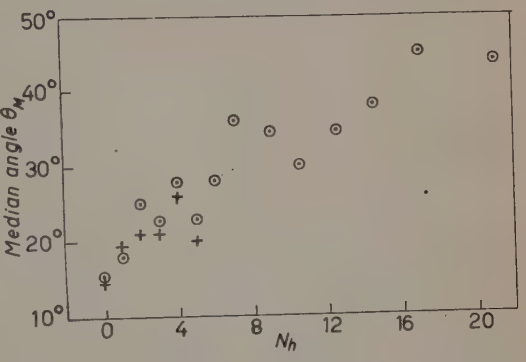

Fig. 8. – Median angle θ_M of all shower particles as a function of N_h. The circles and crosses denote all stars and class A stars respectively.

The above two arguments indicate that the characteristics of class A and class B stars are very similar and in these events there is no appreciable contribution from secondary interactions in the same target nucleus. On this basis we consider class A stars to be rich in proton-nucleon interactions and have used them for observations on the energy and angular distribution of secondary protons and pions. We have selected from class A stars those of the type $0+0+n_s$ and $0+1+n_s$ (referred to as class B) for i) investigating the true multiplicity in nucleon-nucleon collisions and ii) to deduce information regarding the backward recoiling proton in the C.M. system from the « grey » protons. In Section 3 of this paper, we have given reasons to show that the stars included in this group are almost all pure proton-nucleon collisions.

Appendix II

« Spurious scattering » in the emulsions used.

Before attempting to estimate the energies of secondary relativistic particles (of energy up to ~ 6 GeV) from multiple Coulomb scattering measurements, it is important to know the inherent « spurious scattering » present in the emulsions used. For this, extensive measurements were made on the beam protons using a Koristka microscope. The dip angle δ of the beam protons with respect to the plane of the emulsion was such that $\mathrm{ctg}\,\delta \approx 100$.

The variation of « spurious scattering » as a function of emulsion depth was studied in the manner described by Biswas *et al.* ([21]). We found that within errors, it was constant from about 50 μm of the air surface of the emulsion up to about 50 μm from the glass surface; this central region of thickness 500 μm in the emulsion is referred to as the « useful region »; thicknesses are given with reference to the unprocessed emulsion. These emulsions have

([21]) S. Biswas, B. Peters and Rama: *Proc. Ind. Acad. Sci.*, **41** A, 154 (1955).

very low « spurious scattering »; the mean second differences corresponding to « spurious scattering » alone are 0.13 μm, 0.15 μm and 0.26 μm for cell sizes of 500 μm, 1000 μm and 2000 μm, respectively. These values are based on measurements made on a total track length of about one metre.

Appendix III

Identification of relativistic pions and protons.

In the use of the multiple scattering *vs.* grain density method for identifying relativistic protons and pions, it can be seen, from the curves of energy loss *vs.* energy for such particles traversing nuclear emulsions, that:

i) for $p\beta$ values less than about 1.5 GeV/c the identification is unambiguous and easy;

ii) for $p\beta$ values between about 1.5 and 2.5 GeV/c identification is, in general, not possible;

iii) for $p\beta$ values between about 2.5 and 7 GeV/c identification is possible, (if K-mesons are negligible in number compared to pions or protons) but requires great precision of measurement, particularly of the grain density.

In order to extend the useful measurable energy region right upto 6 GeV, we took the following precautions regarding the selection of tracks and measurements of multiple scattering and of grain density.

Selection of tracks:

i) For tracks emitted at angles $\leqslant 10°$ with respect to the primary, the « useful » length should be $\geqslant 6$ mm in that emulsion ($\operatorname{ctg}\delta \geqslant 20$).

ii) For tracks at angles $> 10°$, the « useful » length should be $\geqslant 3$ mm in that emulsion ($\operatorname{ctg}\delta \geqslant 10$).

The greater length criterion for tracks at angles $\leqslant 10°$ was adopted because, as will be seen later, almost all the high energy secondary particles ($p\beta > 2.5$ GeV/c) are emitted within this angle.

Multiple scattering measurements. – All scattering measurements were made on a Koristka microscope. The measurements were made using primary cell sizes of 500 μm for tracks with angles $\leqslant 10°$ and 250 μm for those at larger angles. Measurements were restricted to the « useful » regions of the emulsions as stated in Appendix II. Corrections were then applied for the following:

i) Reading, grain and stage noises associated with scattering measurements which were estimated by the method described by Biswas *et al.* ([21]).

ii) Spurious scattering which was determined as described in Appendix II.

Mean angles of scattering were usually determined from two cell sizes which had mean deviations $\geqslant 2$ times the total noise (including the spurious scattering). We then calculated the $p\beta$ values using the scattering constants given by BISWAS *et al.* ([22]).

We have determined the spurious scattering for the beam protons (ctg $\delta \approx 100$) and used it for correcting the mean deviations measured for the selected tracks of shower particles. It is quite possible that the spurious scattering increases with increasing steepness of the track. In order to study this effect we analyzed the relative numbers of particles in different energy intervals as a function of the dip angle for tracks ejected at angles $\leqslant 10°$. For particles with $2 < p\beta < 4$ GeV/c there is a detectable but small contribution when the tracks have ctg $\delta < 50$, while for particles with $p\beta > 4$ GeV/c this effect is apparent for values of ctg δ up to 100. Thus the extent to which we underestimate the energy depends both on the energy of the particle and its dip angle.

Grain density measurement:

i) Personal errors: All grain counting measurements were made by a single observer. Tracks were always fed to the observer for counting by another person. About 2000 grains were counted on a standard track every alternate day and it was found that the standard error due to personal errors was 1.5%.

ii) Sensitivity variation: No measurements were made in the top 50 μm and the bottom 50 μm of the unprocessed emulsion. The rest of the emulsion showed a linear decrease of sensitivity with depth. This was determined accurately by counting a total of about 20000 grains on tracks of beam protons. The maximum variation with respect to the middle of the emulsion is $\pm 4\%$. This correction for any depth is known with an accuracy better than 1%. Further, there was no detectable variation of grain density($\ll 2\%$) from one region of the emulsion to another.

iii) Statistical errors: A total of about 1800 grains was counted on each track in the case of tracks observed within an angle of 10° with respect to the primary direction and about 700 grains were counted on each of the remaining tracks.

Results. – The results of measurements of scattering and grain density on a total of 114 tracks is shown in Fig. 9. We have also plotted in this figure the grain densities in the tracks of beam protons and of electrons with energies between 20 and 35 MeV from μ-e decays. 9000 grains and 20000 grains were counted on the electron and the 6.2 GeV proton tracks respectively. These two grain densities are, therefore, known with an accuracy better than 2%; the values are 25.6 and 22.8 grains per 100 μm, respectively.

For grain densities between 40 and 23 grains per 100 μm the curve drawn agrees well with that of the Bristol group ([20]). There is, however, some un-

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([22]) S. BISWAS, N. DURGA PRASAD and S. MITRA: *Proc. Ind. Acad. Sci.*, **46** A, 167 (1957).

certainty still about the relativistic rise of grain density ([20,23]). The grain density of the 6.2 GeV proton tracks is 12% lower than that for the electrons from μ-e decay in our case, whereas in the Bristol curve this difference is only about 1% or less. It seems, therefore, that their $p\beta$-grain density curve in the region of the relativistic increase, rises more rapidly than what we find from our measurements. For this work, we have drawn a straight line joining the values of grain density for the 6.2 GeV protons and μ-e decay electrons.

From Fig. 9, it can be seen that four values of $p\beta > 2.5$ GeV/c, it is possible to identify the pions and the protons. The fact that it is found possible

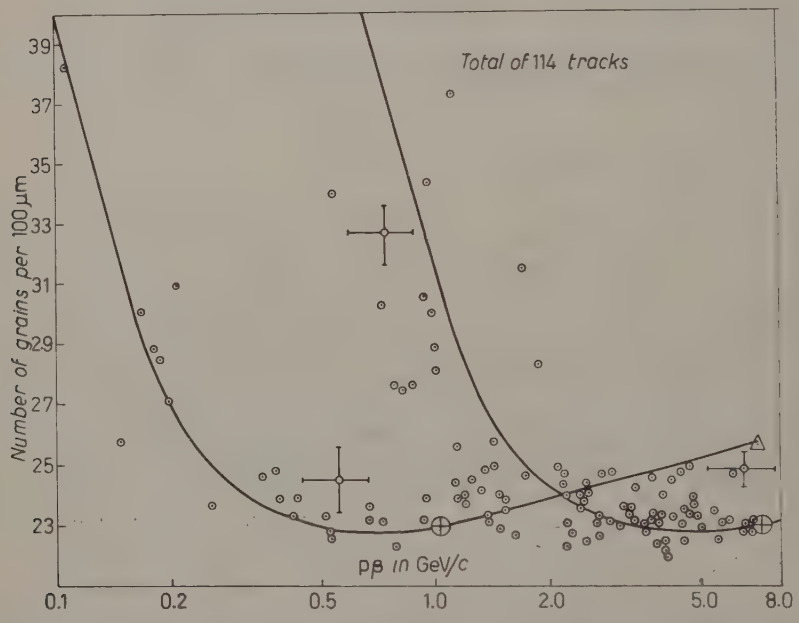

Fig. 9. – Plot of grain density as a function of $p\beta$ for all selected tracks from class A stars. The points marked ⊕ correspond to the grain density of the 6.2 GeV proton tracks and the one marked △ to that of the electrons from selected μ-e decay events. The different magnitudes of mean errors arising from different selection criteria for low and high energy particles are indicated on three points.

to identify such particles, in spite of the increasing contribution of spurious scattering is because the grain density — $p\beta$ curve, in this region of $p\beta$, is insensitive to the errors in $p\beta$. For particles with $p\beta < 1.5$ GeV/c, the identification is straightforward. For those with $p\beta$ between 1.5 and 2.5 GeV/c the identification was made in the following manner. The energies and angles of emission of these particles in the C.M. system were calculated assuming

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([23]) B. Stiller and M. M. Shapiro: *Phys. Rev.*, **92**, 735 (1953).

them to be protons and mesons and in the majority of cases from consistency arguments the particles could be identified. The number of cases where an erroneous identification is made in this manner is small.

RIASSUNTO (*)

Si sono studiate le interazioni nucleari prodotte da protoni di energia 6.2 GeV nelle emulsioni nucleari per dedurre informazioni concernenti alcune delle caratteristiche delle collisioni protone-nucleone. Si sono rilevate 703 disintegrazioni con lo scanning « lungo la traccia ». Si sono eseguite accurate misure dello scattering multiplo e della densità di grani su opportune tracce di particelle relativistiche secondarie; con l'uso di queste misure è stato possibile identificare le particelle sino alle più elevate energie coinvolte. In questo studio si è trovato che in collisioni di protoni di energia di 6.2 GeV con nucleoni: *a*) la molteplicità media del pione carico è 1.51 ± 0.18; *b*) l'anelasticità media è 0.43, ed è, entro i limiti d'errore, indipendente dalla molteplicità totale n_s; *c*) i nucleoni che rinculano nel sistema del C.M. sono fortissimamente collimati nelle direzioni anteriore e posteriore; *d*) i mesoni creati nella collisione presentano pure un'apprezzabile collimazione antero-posteriore nel sistema del C.M.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVI, N. 1 1° Aprile 1960

On the Quasi-Static Approximation in General Relativity (*).

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Department of Physics, Syracuse University - Syracuse, N.Y.

(ricevuto il 30 Novembre 1959)

Summary. — The three main methods used for solving the quasi-static field equations are discussed and it is shown that certain theorems have to be proved before it can be said that the physical equations of motion follow from the symmetry of the field around the sources. We have proved these results, and in the process have shown that there are seven and only seven physical equations of motion for each particle. These correspond to the classical equations of energy, motion and angular momentum. Also, we have shown that the quasi-static field equations may be integrated without expanding any particle parameters, and without introducing a stress-energy tensor.

1. – Introduction.

In a previous series of papers (1), we have developed the theory of the « fast particle » approximation for calculating the field around, and the equations of motion of, an arbitrary particle in general relativity. In particular, we found that it is not necessary to expand the lower order parameters, such as the mass, dipole moment, and angular momentum vector, in order to solve the field equations by successive approximation. Instead, the parameters introduced into the first approximation were the physical parameters of the particle, and not those of the first approximation as in the usual methods.

In this paper, we shall discuss the quasi-static approximation procedure which was first introduced by Einstein, Infeld and Hoffmann (**) in 1938 (2). We shall find that in all the methods which have been used to deduce the equa-

(*) This work was supported by Air Force contract no. AF33 (616)-5069.

(**) Hereafter refered to as EIH.

(1) R. Kerr: I, II and III, *Nuovo Cimento*, **13**, 469, 492 and 673 (1959).

(2) A. Einstein, L. Infeld and B. Hoffmann: *Ann. Math.*, **39**, 65 (1938).

tions of motion at least one of a group of allied theorems must be proved before it can be said that the equations of motion follow from the quasi-symmetry of the field around the particle.

In the second Section we shall give a brief discussion of the quasi-static expansion, showing how it is associated with a non-physical λ-dependent coordinate transformation. In the third, fourth and fifth section we shall discuss the three main groups of methods used to solve the set of approximation equations obtained by the quasi static expansion, while in the sixth section we shall endeavour to combine the best of each.

In the original EIH method the r'th order field equations were to be solved by introducing a non-classical source function, $\mathfrak{T}^{\mu\nu}$, on the right hand side of the field equations,

$$\mathfrak{G}^{\mu\nu} = -8\pi\mathfrak{T}^{\mu\nu}, \tag{1.1}$$

instead of solving the usual field equations,

$$G^{\mu\nu} = 0. \tag{1.2}$$

It was suggested that after the integration of the field equations had been carried out, a solution of the Einstein equations, would be obtained by setting $\mathfrak{T}^{\mu\nu}$ to zero. The second method used to calculate the equations of motion was discovered by EIH in 1949 ([3]). They solved the quasi-static field equations by introducing a mass, dipole moment and angular momentum vector into each approximation.

The third method used to solve the field equations was introduced by FOCK ([4]), and subsequently extended by PAPAPETROU ([5]). By using the De Donder coordinate conditions,

$$(\sqrt{-g}\cdot g^{\mu\nu})_{,\nu} = 0, \tag{1.3}$$

and a *localized* source function in (1.1), they were able to calculate the equations of motion in the sixth approximation. Unlike the EIH solution, their field was non-singular at the particles.

The most important feature of the EIH procedures was that it was found that certain consistency relations must hold in each approximation before the field equations can be solved in the higher orders. By expanding the particle parameters, EIH ([3]) showed that these conditions were equivalent to a set

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([3]) A. EINSTEIN and L. INFELD: *Can. Journ. Math.*, **1**, 209 (1949).
([4]) V. FOCK: *Žurn. Èksp. Teor. Fiz.*, **1**, 81 (1939).
([5]) A. PAPAPETROU: *Proc. Phys. Soc.* (*London*), **64**, 57 (1951).

of differential equations. However, since there was a finite set of equations to be solved *in each approximation*, the individual r'th order equations were non-physical. EIH attempted to show that the differential equations satisfied by the *total* dipole moment, etc.,

$$D^n = \sum_r \underset{r}{D^n},$$

where $\underset{r}{D^n}$ is the r'th order dipole moment, depended on the *physical* parameters alone, and not on their individual r'th order components. As was shown in a paper by MOFFAT and KERR ([6]), their proof was incorrect.

In the Fock-Papapetrou method, the use of a co-ordinate condition reduces the field equations to Poisson equations which may always be solved. However, once a solution for $g_{\mu\nu}$ is obtained it is necessary to prove that the co-ordinate conditions are satisfied. In other words, it is necessary to prove that the divergence relations which they imposed on their source function,

$$\mathfrak{T}^{\mu\nu}{}_{;\nu} = 0,$$

imply that the De Donder conditions are satisfied. Otherwise their solution is not a solution of the original Einstein equations to the required approximation. We shall discuss this in Section **5**.

There are several other unsolved problems in the EIH method as well as that of deducing the equations of motion from the field equations. The first of these is the question of the convergence of the solution. So far, very little is known about the convergence of series solutions for non-linear partial differential equations in more than two variables, so it appears that this problem is going to remain unsolved for a long time. Another problem is the dependence of the solution on the boundary conditions at infinity. In the EIH solution it has been usual to assume that the field is asymptotically Galilean at infinity. However, it is now well known that in the higher orders no such solution exists. It is not known what boundary conditions should be imposed on the r'th order field, $\underset{r}{g}_{\mu\nu}$, so that the limiting field, $g_{\mu\nu}$, will be Galilean at spatial infinity, even when the $\underset{r}{g}_{\mu\nu}$ are not.

The third unsolved problem is what is meant by a « spherically symmetric » particle. There are an infinity of degrees of freedom for a particle in general relativity ([7]) so that it is necessary to distinguish between those which are to be called spherically symmetric and those which are not. For a single source in space there is a solution which is spherically symmetric at the particle, but

([6]) J. MOFFAT and R. KERR: unpublished (reprints available).

([7]) J. BOARDMAN and P. G. BERGMANN: *Phys. Rev.*, **115**, 1318 (1959).

if there are several particles in the field there does not exist a coordinate system with this property.

In this paper we shall not consider the convergence or the dependence on the boundary particles of the EIH solution. We shall just consider the derivation of the physical equations of motion from the field at the particles. In Section 6 we shall see how the equations of motion may be derived without having to expand the particle parameters, without assuming any coordinate conditions, and without introducing a source tensor, whether non-localised as in the EIH solution of Section 3, or localised as in the Fock-Papapetrou procedure.

2. – The quasi-static expansion.

It is comparatively easy to calculate the equations of motion in the fast approximation to the second order, but, because of the difficulty of integrating the functions involved, it is difficult to go beyond this to the post-Newtonian approximation. In the EIH method, however, the forces for slow moving particles may be obtained to any desired order.

In the quasi-static approximation the field is expanded as a power series in an « indeterminate parameter », λ. As we have already mentioned in an earlier paper [1], there is no need to give any physical interpretation to λ, apart from that of a weighting parameter. EIH also assumed that the field was slowly varying in time, so that the derivatives with respect to the non-covariant time co-ordinate, t or x^0, may be considered small compared to those with respect to the spatial coordinates. In order to give this assumption some mathematical form, they introduced a new co-ordinate, τ, which was defined by $\tau = \lambda^{-1}x^0$. In the « co-ordinate system » (τ, x^n), the field is written as

$$\overset{*}{\mathfrak{g}}{}^{\mu\nu}(\lambda) = \eta^{\mu\nu} + \sum_r \lambda^r \overset{r}{h}{}^{\mu\nu}(x^0, x^n)\,, \tag{2.1}$$

(*)

where $\overset{*}{\mathfrak{g}}{}^{\mu\nu}(\lambda)$ and $\mathfrak{g}^{\mu\nu}(\lambda)$ are the fundamental, λ-dependent, contravariant tensor densities in the coordinate system (τ, x^n) and (x^0, x^n), respectively. $\eta^{\mu\nu}$ is the Galilean metric tensor with signature $(-, +, +, +)$, and $\overset{r}{h}{}^{\mu\nu}(x^\varkappa)$ is of the r'th approximation order.

If we change to coordinate (x^0, x^n), we have

$$\begin{cases} \mathfrak{g}^{00}(\lambda) &= \lambda^{-1}\Big(-1 + \sum_r \lambda^r \overset{r}{h}{}^{00}(x^\alpha)\Big)\,, \\ \mathfrak{g}^{0n}(\lambda) &= \sum_r \lambda^r \overset{r}{h}{}^{0n}(x^\alpha)\,, \\ \mathfrak{g}^{mn}(\lambda) &= \lambda^{-1}\Big(\delta^{mn} + \sum_r \lambda^r \overset{r}{h}{}^{mn}(x^\alpha)\Big)\,, \end{cases} \tag{2.2}$$

(*) Greek indices go from 0 to 3, Latin indices from 1 to 3.

since $\mathfrak{g}^{\mu\nu}$ transforms as a tensor density under coordinate transformations. We shall now expand the Einstein tensor density, $\mathfrak{G}^{\mu\nu}$, in powers of λ. Because of its transformation properties, we have

$$\mathfrak{G}^{00}(\lambda) = \lambda\mathring{\mathfrak{G}}^{00}(\lambda)\,, \qquad \mathfrak{G}^{0n}(\lambda) = \mathring{\mathfrak{G}}^{0n}(\lambda)\,, \qquad \mathfrak{G}^{mn}(\lambda) = \lambda^{-1}\mathring{\mathfrak{G}}^{mn}(\lambda)\,, \tag{2.3}$$

so that we shall obtain the same approximation equations whether we expand $\mathfrak{G}^{\mu\nu}$ or $\mathring{\mathfrak{G}}^{\mu\nu}$. Instead of defining $\underset{r}{\mathfrak{G}}^{\mu\nu}$ as the coefficient of λ^r in $\mathfrak{G}^{\mu\nu}(\lambda)$, we shall define it by

$$\mathring{\mathfrak{G}}^{\mu\nu}(\lambda) = \sum_r \lambda^r \underset{r}{\mathfrak{G}}^{\mu\nu}(x^0, x^n)\,. \tag{2.4}$$

From (2.1), $\underset{r}{h}^{\alpha\beta}$ and its spatial derivatives have weight r in this expansion, but, since

$$\Big(\sum_r \lambda^r \underset{r}{h}^{\mu\nu}(x^0, x^n)\Big)_{,x^\varepsilon \ldots x^t,\, \tau \ldots \tau} = \sum_r \lambda^{r+u} h^{\mu\nu}(x^\alpha)_{,s\ldots t,\, \underset{\leftarrow u \rightarrow}{0 \ldots 0}}\,;$$

we see that when we take the derivative of a function with respect to the time coordinate, x^0, we must raise the weight of the function by unity. This means that in the expansion of the field equations all time derivatives are considered small compared to the spatial derivatives.

In order to obtain a solution of the Einstein equations, $\underset{r}{\mathfrak{G}}^{\mu\nu}$ is set to zero,

$$\underset{r}{\mathfrak{G}}^{\mu\nu} = \underset{r}{\Phi}^{\mu\nu} + \underset{r}{\Lambda}^{\mu\nu} = 0\,, \tag{2.5}$$

where $\underset{r}{\Phi}^{\mu\nu}$ is linear in the spatial derivatives of $\underset{r}{h}^{\alpha\beta}$, and $\underset{r}{\Lambda}^{\mu\nu}$ is a function of weight r in the lower order fields. This set of equations is to be solved by successive approximation and then the *physical* field will be given by

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \sum_r \underset{r}{h}^{\mu\nu}\,, \tag{2.6}$$

i.e. by $\mathfrak{g}^{\mu\nu}(\lambda = 1)$. It should be noticed that $\mathfrak{g}^{\mu\nu}(\lambda)$ is not the physical field since we have only used λ to define the orders of magnitude of the terms in the expansion of $\mathfrak{G}^{\mu\nu}$ as a function of the series in (2.6). Sometimes when a suitable physical constant is present in the differential equations it is convenient to expand in this rather than in an indeterminate parameter. However, it is not necessary that such a parameter should exist.

It has been said that λ should be a « small » parameter, so that the series in (2.2) should converge. However, it is not the size of λ which determines the convergence of $\mathfrak{g}^{\mu\nu}(\lambda)$ but the properties of the field in the lowest approximation, namely $\lambda\underset{1}{h}^{\mu\nu}$. In fact, since λ has the dimensions of a velocity, its magnitude depends on the units chosen.

3. – The original EIH method.

From the Appendix, it may be seen that the $\Phi_r^{\mu\nu}$ are given by

$$-2\Phi_r^{00} \equiv h_r^{00}{}_{,ss} - h_r^{st}{}_{,st} = 2\Lambda_r^{00}\,, \tag{3.1}$$

$$-2\Phi_r^{0n} \equiv h_r^{0n}{}_{,ss} - h_r^{0s}{}_{,sn} = 2\Lambda_r^{0n}\,, \tag{3.2}$$

$$-2\Phi_r^{mn} \equiv h_r^{mn}{}_{,ss} - h_r^{ms}{}_{,sn} - h_r^{ns}{}_{,sm} + \delta^{mn} h_r^{st}{}_{,st} = 2\Lambda_r^{mn}\,. \tag{3.3}$$

In 1938, EIH proved the following lemma,

LEMMA 1. *If $B^{\cdots\overset{\smile}{st}}$ is antisymmetric in s and t, then*

$$\int B^{\cdots\overset{\smile}{st}}{}_{,s}\,\mathrm{d}S_t = 0\,,$$

where the integration is over any closed surface in the three dimensional hyperplane $t = const.$

EIH recognized that

$$\Phi_r^{\mu n} \equiv B_r^{\mu:\overset{\smile}{ns}}{}_{,s}\,, \tag{3.4}$$

where

$$B_r^{0:\overset{\smile}{ns}} = -\tfrac{1}{2}(h_r^{0n}{}_{,s} - h_r^{0s}{}_{,n})\,,$$

$$B_r^{m:\overset{\smile}{ns}} = -\tfrac{1}{2}(h_r^{mn}{}_{,s} - h_r^{ms}{}_{,n}) - \tfrac{1}{2}(\delta^{mn} h_r^{st}{}_{,t} - \delta^{ms} h_r^{nt}{}_{,t})\,.$$

Consequently, from the lemma,

$$\int \Phi_r^{\mu n}\,\mathrm{d}S_n = 0\,, \tag{3.5}$$

which shows that necessary conditions for the existence of a solution of the r'th order field equations are that the following surface integrals should be zero,

$$\mathfrak{C}_r^{\mu} \equiv \frac{1}{2\pi}\int \Lambda_r^{\mu n}\,\mathrm{d}S_n = 0\,, \tag{3.6}$$

for any closed surface on which the field is non-singular.

As well as (3.6), there are a further set of surface integral conditions which must be satisfied before the field equations can be solved, namely

$$\mathfrak{A}_r{}_m = \varepsilon_{mst}\cdot\frac{1}{4\pi}\int x^s \Lambda_r^{tu}\,\mathrm{d}S_u = 0\,. \tag{3.7}$$

To prove this, we observe that if we define

$$\overset{*}{B}{}^{mn}_{rp} = \varepsilon_{pst}\left[x^s\left(\overset{*}{h}{}^{tm}_{r},_{n} - \overset{*}{h}{}^{tn}_{r},_{m} + \delta^{tm}\overset{*}{h}{}^{nu}_{r},_{u} - \delta^{tn}\overset{*}{h}{}^{mu}_{r},_{u}\right) - \delta^{tm}\overset{*}{h}{}^{sn}_{r} + \delta^{tn}\overset{*}{h}{}^{sm}_{r}\right]$$

then

$$\overset{*}{B}{}^{mn}_{p},_{n} = -2\varepsilon_{pst}x^s\overset{*}{\Phi}{}^{tm}_{r}\,.$$

From the lemma, and the r'th order field equations, we obtain (3.7).

We have seen that there are seven surface integrals which have to be satisfied before the r'th order field equations can be solved. It is natural to ask whether there are any more conditions which are independent of these.

THEOREM 1. *Necessary and sufficient conditions for the integration of the field equations in the r'th approximation,*

$$\overset{*}{\Phi}{}^{\mu\nu}_{r} + \overset{*}{\Lambda}{}^{\mu\nu}_{r} = 0\,,$$

are that the seven surface integrals, $\overset{*}{\mathfrak{C}}{}^{\mu}_{r}$ *and* $\overset{*}{\mathfrak{A}}{}_{m}$ *should be zero for an* arbitrary *closed surface of integration on which the field is non-singular.*

We have already seen that these conditions are necessary. We shall now show that they are sufficient. First of all, we observe that since $\overset{*}{\mathfrak{C}}{}^{\mu}_{r}$ is zero for an arbitrary surface, the divergence of the integrand must be zero for every non-singular point of the field,

$$\overset{*}{\Lambda}{}^{\mu n}_{r},_{n} = 0\,. \tag{3.8}$$

This follows from Gauss' theorem. Similarly, the divergence of the integrand of (3.7) must be zero, but this just gives equation (3.8) over again.

Let us suppose that there exists a solution, $\overset{*}{h}{}^{\mu\nu}_{r}$, of (3.1) through (3.3). Let $\overset{*}{a}{}^{\mu}_{r}$ be a solution of the Poisson equation,

$$\overset{*}{a}{}^{\mu}_{r},_{ss} = \overset{*}{h}{}^{\mu s}_{r},_{s}\,, \tag{3.9}$$

and let

$$\begin{cases} \overset{*}{h}{}^{00}_{r} = \overset{*}{h}{}^{00}_{r} - \overset{*}{a}{}^{s}_{r},_{s}\,, \\ \overset{*}{h}{}^{0n}_{r} = \overset{*}{h}{}^{0n}_{r} - \overset{*}{a}{}^{0}_{r},_{n}\,, \\ \overset{*}{h}{}^{mn}_{r} = \overset{*}{h}{}^{mn}_{r} - \overset{*}{a}{}^{m}_{r},_{n} - \overset{*}{a}{}^{n}_{r},_{m} + \delta^{mn}\overset{*}{a}{}^{s}_{r},_{s}\,. \end{cases} \tag{3.10}$$

From (3.1) through (3.3) and (3.10), it is easily shown that

$$\overset{*}{h}{}^{\mu\nu},_{ss} = 2\overset{*}{\Lambda}{}^{\mu\nu}_{r}\,, \tag{3.11}$$

$$\overset{*}{h}{}^{\mu n},_{n} = 0\,. \tag{3.12}$$

LEMMA 2. *Necessary and sufficient conditions for the integrability of the field equations in the r'th approximation are that the set of equation,* (3.11) *and* (3.12), *are solvable.*

We have already proved in the last paragraph that these conditions are necessary, since if there exists an $\overset{r}{h}{}^{\mu\nu}$ satisfying (3.1) through (3.3) then $\overset{*}{\overset{r}{h}}{}^{\mu\nu}$ defined by (3.9) and (3.10) will satisfy (3.11) and (3.12). On the other hand, if there exists an $\overset{*}{\overset{r}{h}}{}^{\mu\nu}$ satisfying (3.11) and (3.12) then it will also satisfy the Einstein field equations (3.1) through (3.3).

This shows that we only have to prove that the surface integral conditions are sufficient for the existence of a solution of (3.11) and (3.12) to complete the proof of Theorem 1. Let $\overset{r}{h}{}^{\mu\nu}$ be a particular solution of the Poisson equations, (3.11), which may always be solved. If we take the divergence of (3.11) we see that

$$(\bar{\overset{r}{h}}{}^{\mu n}{}_{,n})_{,ss} = 2\overset{r}{\Lambda}{}^{\mu n}{}_{,n} . \tag{3.13}$$

Since we are now assuming that the surface integral conditions are satisfied, the right hand side of this equation must be zero, and so $\bar{\overset{r}{h}}{}^{\mu n}{}_{,n}$ must be a harmonic function,

$$\bar{\overset{r}{h}}{}^{\mu n}{}_{,n} = \overset{r}{\varphi}{}^{\mu} + \sum_{i,t} \overset{i}{\overset{r}{k}}{}^{\mu:(s)_t} \overset{i}{\Psi}_{,(s)_t} , \tag{3.14 (*)}$$

where $\overset{r}{\varphi}{}^{\mu}$ is a harmonic function which is non-singular at all finite points of the space but is singular at infinity, and

$$\overset{i}{\Psi} = |\overset{i}{r}|^{-1} , \qquad \overset{i}{x}{}^{n} = x^n - \overset{i}{\xi}{}^{n} , \qquad \overset{i}{r} = (\overset{i}{x}{}^{s}\overset{i}{x}{}^{s})^{\frac{1}{2}} .$$

The $\overset{i}{\overset{r}{k}}{}^{\mu:(s)_t}$, which are completely symmetric in the indices s_i, depend on the particular solution of (3.11) chosen.

The reason that $\overset{r}{\varphi}{}^{\mu}$ appears in (3.14) is that, as we have already pointed out, $\overset{r}{h}{}^{\mu\nu}$ cannot vanish at infinity in the higher approximations. This non-singular function will now be shown to have no physical significance. We define $\overset{r}{h}{}^{\mu\nu}$ by

$$\overset{r}{h}{}^{00} = \bar{\overset{r}{h}}{}^{00} , \tag{3.15}$$

$$\overset{r}{h}{}^{0n} = \bar{\overset{r}{h}}{}^{0n} + \overset{r}{\chi}{}^{0n} - \sum_{i,t} \overset{i}{\overset{r}{k}}{}^{0:n(s)_t} \overset{i}{\Psi}_{,(s)_t} , \tag{3.16}$$

$$\overset{r}{h}{}^{mn} = \bar{\overset{r}{h}}{}^{mn} + \overset{r}{\chi}{}^{mn} - \tfrac{1}{2} \sum_{i} (\overset{i}{\overset{r}{k}}{}^{m:n} + \overset{i}{\overset{r}{k}}{}^{m:n}) \overset{i}{\Psi} - \sum_{i,t} (\overset{i}{\overset{r}{k}}{}^{m:nu(s)_t} + \overset{i}{\overset{r}{k}}{}^{n:mu(s)_t} - \overset{i}{\overset{r}{k}}{}^{u:mn(s)_t}) \overset{i}{\Psi}_{,u,(s)_t} , \tag{3.17}$$

(*) We shall use the following notation, $A^{\cdots(s)_t} = A^{\cdots s_1 s_2 \cdots s_t}$. Throughout this paper we shall assume that there are singularities, or particles, at the points with coordinates $\overset{i}{\xi}{}^{n}$.

where $\underset{r}{\chi}^{\mu\nu}$ is the following function,

$$\underset{r}{\chi}^{01} = -\int_0^{x^1} \underset{r}{\varphi}^0(\alpha, x^2, x^3)\,\mathrm{d}\alpha + \underset{r}{H}(x^2, x^3)\,,$$

$$\underset{r}{\chi}^{02} = \underset{r}{\chi}^{03} = 0\,,$$

$$\chi^{mn} = 0\,, \qquad \text{for } m \neq n,$$

$$\underset{r}{\chi}^{11} = -\int_0^{x^1} \underset{r}{\varphi}^1(\alpha, x^2, x^3)\,\mathrm{d}\alpha + \underset{r}{F}^1(x^2, x^3)\,,$$

and similarly for $\underset{r}{\chi}^{22}$ and $\underset{r}{\chi}^{33}$ with all indices interchanged cyclically. It may be shown that $\underset{r}{\chi}^{\mu\nu}$ will be a harmonic function, provided that

$$\text{(3.18)} \qquad \begin{cases} \underset{r}{H}(x^2, x^3)_{,ss} = \underset{r}{\varphi}^0{}_{,1}(0, x^2, x^3)\,, \\ \underset{r}{F}^1(x^2, x^3)_{,ss} = \underset{r}{\varphi}^1{}_{,1}(0, x^2, x^3)\,, \text{ etc.} \end{cases}$$

Also,

$$\text{(3.19)} \qquad \underset{r}{\chi}^{\mu m}{}_{,n} = -\underset{r}{\varphi}^{\mu}(x^1, x^2, x^3)\,.$$

Since (3.18) may always be solved, being Poisson equations, we can always find a non-singular function which is harmonic and satisfies (3.19). From (3.14) through (3.17), and (3.19), we see that

$$\text{(3.20)} \qquad \underset{r}{h}^{0n}{}_{,n} = \sum_i \overset{i}{\underset{r}{k}}{}^0 \overset{i}{\Psi}\,,$$

$$\text{(3.21)} \qquad \underset{r}{h}^{mn}{}_{,n} = \sum_i (\overset{i}{\underset{r}{k}}{}^m \overset{i}{\Psi} + \overset{i}{\underset{r}{k}}{}^{\underset{\sim}{mn}} \overset{i}{\Psi}_{,n})\,,$$

where $\overset{i}{\underset{r}{k}}{}^{\underset{\sim}{mn}}$ is the antisymmetric part of $\overset{i}{\underset{r}{k}}{}^{m:n}$,

$$\text{(3.22)} \qquad \overset{i}{\underset{r}{k}}{}^{\underset{\sim}{mn}} = \tfrac{1}{2}(\overset{i}{\underset{r}{k}}{}^{m:n} - \overset{i}{\underset{r}{k}}{}^{n:m})\,.$$

We shall now show that the $\overset{i}{\underset{r}{k}}{}^{\mu}$ and $\overset{i}{\underset{r}{k}}{}^{\underset{\sim}{\nu n}}$ are invariants of the differential equations, (3.11), and do not depend on the particular solution, $\underset{r}{\bar{h}}^{\mu\nu}$. To prove this, we shall show that if ${}_a h^{\mu\nu}$, $a = 1, 2$, are any two solutions of (3.11).

$${}_a h^{\mu\nu}{}_{,ss} = 2\Lambda^{\mu\nu}\,,$$

such that

$$ {}_a h^{\mu n}{}_{,n} = \sum_i ({}_a \overset{i}{k}{}^{\mu} \overset{i}{\Psi} + {}_a \overset{i}{k}{}^{\mu n} \overset{i}{\Psi}_{,n}) , $$

then

$$ (3.23)\ (*) \qquad \begin{cases} \overset{i}{K}{}^{\mu} = {}_1\overset{i}{k}{}^{\mu} - {}_2\overset{i}{k}{}^{\mu} = 0 , \\ \overset{i}{K}{}^{mn} = {}_1\overset{i}{k}{}^{mn} - {}_2\overset{i}{k}{}^{mn} = 0 . \end{cases} $$

If we define $\varrho^{\mu\nu} = {}_1h^{\mu\nu} - {}_2h^{\mu\nu}$, then it satisfies the equations,

$$ (3.24) \qquad \begin{cases} \varrho^{\mu\nu}{}_{,ss} = 0 , \\ \varrho^{\mu n}{}_{,n} = \sum_i (\overset{i}{K}{}^{\mu} \overset{i}{\Psi} + \overset{i}{K}{}^{\mu n} \overset{i}{\Psi}_{,n}) . \end{cases} $$

From the surface integral lemma,

$$ \int^{i} (\varrho^{\mu s}{}_{,t} - \varrho^{\mu t}{}_{,s})_{,s} \, \mathrm{d}S_t = 0 = \sum_j \int^{i} (\overset{j}{K}{}^{\mu} \overset{j}{\Psi} + \overset{j}{K}{}^{\mu n} \overset{j}{\Psi}_{,n})_{,t} \, \mathrm{d}S_t = 4\pi \cdot \overset{i}{K}{}^{\mu} . $$

The i over the integral sign denotes that the integration is over a closed surface surrounding the i'th singularity, but no other. In a completely analogous fashion, we prove that $\overset{i}{K}{}^{mn} = 0$. To do this, we use the surface integral lemma for the $\overset{i}{B}{}^{mn}_{p}$ defined in the equation following (3.7), with $\overset{i}{h}{}^{\mu\nu}$ replaced by $\varrho^{\mu\nu}$. This complete the proof of Lemma 3.

LEMMA 3. *Necessary and sufficient conditions for the integrability of equations* (3.11) *and* (3.12) *are that* $\overset{i}{\Lambda}{}^{\mu n}{}_{,n} = 0$, *and*

$$ \overset{i}{k}{}^{\mu} = 0, \qquad \overset{i}{k}{}^{mn} = 0, $$

where the $\overset{i}{k}{}^{\cdots}$ *are invariants of* (3.11).

We shall now show how the k's are related to the surface integrals, $\overset{i}{\mathfrak{C}}{}^{\mu}$ and $\overset{i}{\mathfrak{A}}_{m}$. As a particular solution of the field equations, (3.11), we take

$$ (3.25) \qquad \overset{i}{\bar{h}}{}^{\mu\nu} = \frac{1}{2\pi} \int_V \frac{\overset{i}{\Lambda}{}^{\mu\nu}(r')}{|r - r'|} \, \mathrm{d}^3x' , $$

(*) For the sake of simplicity, we shall omit the approximation order index, r, in this proof. Also, we shall define $k^{\alpha\gamma} = 0$, so that we do not have to write (3.24) as two equations, one for $\mu = 0$ and one for $\mu = m$.

where V is the region enclosed by the surfaces at the particles, S_i, and the surface « at infinity », S. From (3.25),

$$2\bar{\underset{r}{h}}^{\mu n}{}_{,n} = \sum_i \int \overset{i}{\underset{r}{\Lambda}}{}^{\mu n}(r')\,|r-r'|^{-1}\,\mathrm{d}S'_n - \int_S \overset{i}{\underset{r}{\Lambda}}{}^{\mu n}(r')\,|r-r'|^{-1}\,\mathrm{d}S'_n\,, \tag{3.26}$$

where we have integrated by parts, and used $\overset{i}{\underset{r}{\Lambda}}{}^{\mu n}{}_{,n} = 0$. The second term on the right hand side of (3.26) is a non-singular harmonic function inside V, and corresponds to the $\overset{i}{\underset{r}{\varphi}}{}^{\mu}$ of (3.14). If we expand $|r-r'|^{-1}$ about $\overset{i}{\xi}{}^{n}$,

$$\frac{1}{|r-r'|} = \sum_t \frac{(-)^t}{t!}\,\overset{i}{x}{}'(s)_t\,\overset{i}{\Psi}_{,(s)_t}\,, \tag{3.27}$$

then

$$\int \frac{\overset{i}{\underset{r}{\Lambda}}{}^{\mu n}(r')}{|r-r'|}\,\mathrm{d}S_n = \sum_t \frac{(-)^t}{t!}\int \overset{i}{\underset{r}{\Lambda}}{}^{\mu n}(r')\,\overset{i}{x}{}'(s)_t\,\mathrm{d}S_n\cdot\overset{i}{\Psi}_{,(s)_t}\,. \tag{3.28}$$

This expansion is valid outside the S_i. From (3.14), (3.26) and (3.28),

$$\overset{i}{\underset{r}{k}}{}^{\mu} = \frac{1}{2\pi}\int \overset{i}{\underset{r}{\Lambda}}{}^{\mu n}(r')\,\mathrm{d}S_n = \overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu}\,,$$

$$\overset{i}{\underset{r}{k}}{}^{mn} = -\frac{1}{4\pi}\int (\overset{i}{\underset{r}{\Lambda}}{}^{ms}\overset{i}{x}{}^{n} - \overset{i}{\underset{r}{\Lambda}}{}^{ns}\overset{i}{x}{}^{m})\,\mathrm{d}S_s = \varepsilon^{mnsi}\overset{i}{\underset{r}{\mathfrak{A}}}{}_{s}\,.$$

Lemma 4. The invariants $\overset{i}{\underset{r}{k}}{}^{\mu}$ and $\overset{i}{\underset{r}{k}}{}^{mn}$ of the differential equations,

$$\underset{r}{h}{}^{\mu\nu}{}_{,ss} = 2\overset{i}{\underset{r}{\Lambda}}{}^{\mu\nu}\,, \qquad \underset{r}{h}{}^{\mu n}{}_{,n} = 0\,, \qquad \overset{i}{\underset{r}{\Lambda}}{}^{\mu n}{}_{,n} = 0\,,$$

are related to the surface integrals by the following equations,

$$\overset{i}{\underset{r}{k}}{}^{\mu} = \overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu}\,,$$
$$\overset{i}{\underset{r}{k}}{}^{mn} = \varepsilon^{mnsi}\overset{i}{\underset{r}{\mathfrak{A}}}{}_{s}\,,$$

From Lemmas 2 through 4, we see that necessary and sufficient conditions for the integrability of the Einstein field equations in the r'th approximation are that the surface integrals, $\overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu}$ and $\overset{i}{\underset{r}{\mathfrak{A}}}{}_{m}$ should be zero and that $\overset{i}{\underset{r}{\Lambda}}{}^{\mu n}{}_{,n} = 0$. This completes the proof of Theorem 1, since these conditions are equivalent to the seven surface integrals being zero for an arbitrary surface of integration.

In their 1938 paper, EIH gave a proof of a theorem corresponding to Theorem I, but using completely different methods. What they endeavoured to prove was that the *four* surface integral conditions, $\overset{i}{\underset{r}{\mathfrak{G}}}{}^{\mu} = 0$, were sufficient for the integrability of the field equations. This is not true since it is also necessary to have the other three surface integral conditions satisfied. These conditions are independent of the four EIH surface integrals and correspond to the classical equations of angular momentum.

We shall now discuss the method used by EIH in their 1938 and 1940([8]) papers. As we mentioned in the previous paragraph, they did not consider the three $\overset{i}{\underset{r}{\mathfrak{A}}}_m$, which arise because $g_{\mu\nu}$ is a *symmetric* tensor. In our discussion of their procedure we shall extend their analysis to allow for these extra conditions.

In 1940, EINSTEIN and INFELD suggested that a non-localized source function, $\mathfrak{T}^{\mu\nu}$, should be introduced on the right hand side of the field equations,

$$\mathfrak{G}^{\mu\nu} \equiv \Phi^{\mu\nu} + \Lambda^{\mu\nu} = -8\pi\mathfrak{T}^{\mu\nu} . \tag{3.29}$$

The integrations were to be carried out using (3.29) and then a solution of the Einstein equations was to be obtained by taking

$$\mathfrak{T}^{\mu\nu} \equiv \sum_i \mathfrak{T}^{\mu\nu} = 0 . \tag{3.30}$$

As before, necessary and sufficient conditions for the integrability of equations (3.29) are that

$$\begin{cases} \overset{i}{\underset{r}{\mathfrak{G}}}{}^{\mu} = \dfrac{1}{2\pi}\displaystyle\int^{i} (\Lambda^{\mu n} + 8\pi\mathfrak{T}^{\mu n})\, \mathrm{d}S_n = 0 , \\ \overset{i}{\underset{r}{\mathfrak{A}}}_m = \dfrac{1}{4\pi}\displaystyle\int^{i} \varepsilon_{mst}\, \overset{i}{x}{}^{s} (\Lambda^{tu} + 8\pi\mathfrak{T}^{tu})\, \mathrm{d}S_u = 0 , \end{cases} \tag{3.31}$$

$$(\Lambda^{\mu n} + 8\pi\mathfrak{T}^{\mu n})_{,n} = 0 . \tag{3.32}$$

Since the source function has no physical significance in this procedure, it may be chosen as

$$\begin{cases} 8\pi\mathfrak{T}^{00} = 0 , \qquad 8\pi\mathfrak{T}^{0n} = \mathfrak{D}^{0}{}_{,n} , \\ 8\pi\mathfrak{T}^{mn} = \mathfrak{D}^{m}{}_{,n} + \mathfrak{D}^{n}{}_{,m} - \delta^{mn}\mathfrak{D}^{s}{}_{,s} . \end{cases} \tag{3.33}$$

Equation (3.32) will be satisfied if

$$\mathfrak{D}^{\mu}{}_{,ss} = -\Lambda^{\mu n}{}_{,n} . \tag{3.34}$$

([8]) A. EINSTEIN and L. INFELD: *Ann. Math.*, **41**, 797 (1940).

Let $\overset{*}{\underset{r}{\mathfrak{D}}}{}^{\mu}$ be a solution of these equations, and write

$$\overset{}{\underset{r}{\mathfrak{D}}}{}^{\mu} = \overset{*}{\underset{r}{\mathfrak{D}}}{}^{\mu} + \sum_i \left(\overset{i}{\underset{r}{C}}{}^{\mu}\overset{i}{\underset{r}{\Psi}} + \overset{i}{\underset{r}{B}}{}^{\mu n}\overset{i}{\underset{r}{\Psi}}_{,n}\right), \qquad \overset{i}{\underset{r}{B}}{}^{0n}_{\smile} = 0 . \tag{3.35}$$

From the surface integral Lemma,

$$\begin{cases} \int^{i} (\delta^{ms}\overset{}{\underset{r}{\mathfrak{D}}}{}^{n} - \delta^{mn}\overset{}{\underset{r}{\mathfrak{D}}}{}^{s})_{,s}\,\mathrm{d}S_n = 0 , \\ \int^{i} [\varepsilon_{mst}\overset{i}{x}{}^{s}(\delta^{tu}\overset{}{\underset{r}{\mathfrak{D}}}{}^{n} - \delta^{nt}\overset{}{\underset{r}{\mathfrak{D}}}{}^{u})]_{,u}\,\mathrm{d}S_n = 0 . \end{cases} \tag{3.36}$$

From (3.31), (3.33) and (3.36), we have

$$\begin{cases} \overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu} = \dfrac{1}{2\pi}\displaystyle\int^{i} (\overset{}{\underset{r}{\Lambda}}{}^{\mu n} + \overset{}{\underset{r}{\mathfrak{D}}}{}^{\mu}{}_{,n})\,\mathrm{d}S_n , \\ \overset{i}{\underset{r}{\mathfrak{A}}}{}_{m} = \dfrac{1}{4\pi}\displaystyle\int^{i} \varepsilon_{mst}[\overset{i}{x}{}^{s}(\overset{}{\underset{r}{\Lambda}}{}^{tn} + \overset{}{\underset{r}{\mathfrak{D}}}{}^{t}{}_{,n}) - \delta^{ns}\overset{}{\underset{r}{\mathfrak{D}}}{}^{t}]\mathrm{d}S_n , \end{cases} \tag{3.37}$$

where the integration is over the surface enclosing the i'th singularity. From (3.35) and (3.37),

$$\overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu} = \sum_j \int^{i} \frac{1}{2\pi}\left(\overset{j}{\underset{r}{C}}{}^{\mu}\overset{j}{\underset{r}{\Psi}} + \overset{j}{\underset{r}{B}}{}^{\mu n}\overset{j}{\underset{r}{\Psi}}_{,n}\right)_{,s}\mathrm{d}S_s + \frac{1}{2\pi}\int^{i} (\overset{}{\underset{r}{\Lambda}}{}^{\mu n} + \overset{*}{\underset{r}{\mathfrak{D}}}{}^{\mu}{}_{,n})\,\mathrm{d}S_n ,$$

and similarly for $\overset{i}{\underset{r}{\mathfrak{A}}}{}_{m}$. The surface integral conditions will be satisfied for an arbitrary surface, provided that the $\overset{i}{\underset{r}{C}}{}^{\mu}$ and $\overset{i}{\underset{r}{B}}{}^{\mu n}_{\smile}$ satisfy

$$2\,\overset{i}{\underset{r}{C}}{}^{\mu} = +\frac{1}{2\pi}\int^{i} (\overset{}{\underset{r}{\Lambda}}{}^{\mu n} + \overset{*}{\underset{r}{\mathfrak{D}}}{}^{\mu}{}_{,n})\,\mathrm{d}S_n , \tag{3.38}$$

$$2\,\overset{i}{\underset{r}{B}}{}^{mn}_{\smile} = \frac{1}{4\pi}\int^{i} [(\overset{}{\underset{r}{\Lambda}}{}^{ns} + \overset{*}{\underset{r}{\mathfrak{D}}}{}^{n}{}_{,s})\overset{i}{x}{}^{m} - (\overset{}{\underset{r}{\Lambda}}{}^{ms} + \overset{*}{\underset{r}{\mathfrak{D}}}{}^{m}{}_{,s})\overset{i}{x}{}^{n} - \delta^{ms}\overset{*}{\underset{r}{\mathfrak{D}}}{}^{n} + \delta^{ns}\overset{*}{\underset{r}{\mathfrak{D}}}{}^{m}]\,\mathrm{d}S_s . \tag{3.39}$$

This shows that by solving the Poisson equation, (3.34), for $\overset{*}{\underset{r}{\mathfrak{D}}}{}^{\mu}$, and then calculating the surface integrals of (3.38) and (3.39), we can construct a $\overset{}{\underset{r}{\mathfrak{D}}}{}^{\mu}$ which satisfies all the conditions necessary for the integration of the r'th order field equations, (3.29). In order to derive a solution of Einstein's field equations, we have to equate $\mathfrak{T}^{\mu\nu}$, and so $\mathfrak{D}^{\mu}$, to zero after the integrations have

been performed to all orders. Since the source function is a function of position, this apparently gives a set of conditions to be satisfied at every point of the field. If it were possible to take $\overset{*}{\underset{r}{\mathfrak{D}}}{}^{\mu}$ to be zero, as EINSTEIN and INFELD proposed in their 1940 paper, then we would have

$$\mathfrak{D}^{\mu} = \sum_{i} \left(\overset{i}{C}{}^{\mu} \overset{i}{\Psi} + \overset{i}{B}{}^{\mu n}_{\smile} \overset{i}{\Psi}_{,n} \right), \tag{3.40}$$

where

$$\overset{i}{C}{}^{\mu} = \sum_{r} \overset{i}{\underset{r}{C}}{}^{\mu}, \qquad \overset{i}{B}{}^{mn}_{\smile} = \sum_{r} \overset{i}{\underset{r}{B}}{}^{mn}_{\smile}. \tag{3.41}$$

This would mean that we could obtain a solution of the Einstein field equations by equating $\overset{i}{C}{}^{\mu}$ and $\overset{i}{B}{}^{mn}_{\smile}$ to zero, which would give seven equations to be satisfied for each particle.

However, in general $\underset{r}{A}{}^{lmn}_{,n}$ will not be zero in the higher approximations which means that we cannot choose $\overset{*}{\underset{r}{\mathfrak{D}}}{}^{\mu}$ to be zero. This means that $\mathfrak{D}^{\mu}$ will not be a harmonic function and need not be linear in $\overset{i}{C}{}^{\mu}$ and $\overset{i}{B}{}^{mn}_{\smile}$. Fortunately, the following theorem can be proved:

THEOREM II. *Provided that the integrations are carried out consistently,*

$$\overset{i}{C}{}^{\mu}, \ \overset{i}{B}{}^{mn}_{\smile} = 0 \qquad \textit{imply that} \quad \mathfrak{D}^{\mu} = 0,$$

and consequently that $\mathfrak{g}^{\mu\nu}$ *is a solution of the Einstein equations (neglecting all convergence considerations). Furthermore, if we define the partial sums for any parameter or function,* F, *by*

$$\underset{(r)}{F} = \sum_{s=0}^{r} \underset{s}{F}, \tag{3.42}$$

then $\underset{(r)}{\mathfrak{D}}{}^{\mu}$ *will be zero to the r'th approximation whenever* $\overset{i}{\underset{(r)}{C}}{}^{\mu}$ *and* $\overset{i}{\underset{(r)}{B}}{}^{mn}_{\smile}$ *are zero to the same order* (*).

We shall not prove this at present, but shall wait until we have proved Theorem V, from which it follows trivially. Furthermore, the original EIH procedure is a very inefficient way to calculate the field around a set of singularities, since it is necessary to solve (3.34) as well as the field equations, (3.29).

(*) When we say that $\underset{(r)}{F}$ is zero to the r'th approximation, we mean that $\underset{(r)}{F} = \underset{r+1}{f}$, where $\underset{r+1}{f}$ is of the $(r+1)$'th, or higher, order.

4. – The new approximation method of Einstein and Infeld.

In 1949, EINSTEIN and INFELD ([3]) showed that if the contracted Bianchi identities are expanded in the (τ, x^n) co-ordinates of Section 2,

$$\overset{*}{\mathfrak{G}}{}^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\beta}\overset{*}{\mathfrak{G}}{}^{\alpha\beta} \equiv 0 , \tag{4.1}$$

then the coefficient of λ^r must be identically zero,

$$\underset{r}{\mathfrak{G}}{}^{\mu n}{}_{,n} + \underset{r-1}{\mathfrak{G}}{}^{\mu 0}{}_{,0} + \sum_s \underset{s}{\Gamma}{}^{\mu}_{\alpha\beta}\,\underset{r-s}{\mathfrak{G}}{}^{\alpha\beta} \equiv 0 , \tag{4.2}$$

that is

$$\underset{r}{\Phi}{}^{\mu n}{}_{,n} + \underset{r}{\Lambda}{}^{\mu n}{}_{,n} + \underset{r-1}{\mathfrak{G}}{}^{\mu 0}{}_{,0} + \sum_s \underset{s}{\Gamma}{}^{\mu}_{\alpha\beta}\,\underset{r-s}{\mathfrak{G}}{}^{\alpha\beta} \equiv 0 . \tag{4.3}$$

Since $\underset{r}{\Phi}{}^{\mu\nu}$ is a function of $\underset{r}{h}{}^{\alpha\beta}$, alone, and since the remaining terms in (4.3) are independent of the r'th order field, we have

$$\underset{r}{\Phi}{}^{\mu n}{}_{,n} \equiv 0 . \tag{4.4}$$

This may also be verified from equations (3.2) and (3.3). From (4.3) and (4.4),

$$\underset{r}{\Lambda}{}^{\mu n}{}_{,n} \equiv -\underset{r-1}{\mathfrak{G}}{}^{\mu 0}{}_{,0} - \sum_s \underset{s}{\Gamma}{}^{\mu}_{\alpha\beta}\,\underset{r-s}{\mathfrak{G}}{}^{\alpha\beta} . \tag{4.5}$$

This shows that the left hand side is zero whenever the field equations are satisfied in every lower approximation.

We saw in Theorem I that necessary and sufficient conditions for the integrability of the field equations in the r'th approximation are that the seven surface integral conditions should be zero for an arbitrary surface of integration. This is equivalent to

$$\underset{r}{\Lambda}{}^{\mu n}{}_{,n} = 0 , \tag{4.6}$$

and that the surface integrals around the i'th particle should be zero, $\overset{i}{\underset{r}{C}}{}^{\mu} = 0$, etc. From (4.5) we see that (4.6) is satisfied whenever the field equations are satisfied in the lower orders. Consequently, *sufficient*, though not necessary, conditions for the integrability of the r'th order field equations are that

$$\underset{s}{\mathfrak{G}}{}^{\mu\nu} = 0 , \qquad s < r, \tag{4.7}$$

$$\overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu} = 0 , \tag{4.8}$$

$$\overset{i}{\underset{r}{\mathfrak{A}}}{}_{m} = 0 . \tag{4.9}$$

In 1949, EINSTEIN and INFELD showed how the field equations, (3.1) through (3.3), could be satisfied in every approximation. They showed that if $\overset{i}{\underset{r}{h}}{}^{\prime\mu\nu}$ is a solution of the r'th order field equations, then $\overset{i}{\underset{r}{h}}{}^{\mu\nu}$ is also, where

$$\overset{i}{\underset{r}{h}}{}^{00} = \overset{i}{\underset{r}{h}}{}^{\prime 00} + 4\sum_{i}\left(\overset{i}{\underset{r}{m}}\overset{i}{\underset{r}{\Psi}} + \overset{i}{\underset{r}{D}}{}^{s}\overset{i}{\underset{r}{\Psi}}_{,s}\right), \tag{4.10}$$

$$\overset{i}{\underset{r}{h}}{}^{\mu n} = \overset{i}{\underset{r}{h}}{}^{\prime\mu n}, \tag{4.11}$$

where $\overset{i}{\underset{r}{h}}{}^{\prime\mu\nu}$ is independent of $\overset{i}{\underset{r}{m}}$ and $\overset{i}{\underset{r}{D}}{}^{n}$. The first set of surface integrals, (4.8), then have the following form,

$$0 = \overset{i}{\underset{r}{\mathfrak{G}}}{}^{0} = 4\overset{i}{\underset{r-1}{m}}{}_{,0} + \overset{i}{\underset{r}{\mathfrak{G}}}{}^{\prime 0}, \tag{4.12}$$

$$0 = \overset{i}{\underset{r}{\mathfrak{G}}}{}^{n} = 4\overset{i}{\underset{r-2}{m}}\overset{i}{v}{}^{n}{}_{,0} - 4\overset{i}{\underset{r-2}{D}}{}^{n}{}_{,00} + \overset{i}{\underset{r}{\mathfrak{G}}}{}^{\prime n}, \tag{4.13 (*)}$$

where the $\overset{i}{\underset{r}{\mathfrak{G}}}{}^{\prime\mu}$ do not contain any terms *linear* in the $\overset{i}{\underset{r-1}{m}}$ or $\overset{i}{\underset{r-2}{D}}{}^{m}$, and are independent of the higher order parameters. Consequently (4.8) will be satisfied in every approximation if the r'th order masses and dipole moments satisfy (4.12) and (4 13).

As EINSTEIN and INFELD did not admit the existence of the surface integrals in (4.9), they did not attempt to satisfy these conditions in their 1949 paper. The extension of their « new » approximation method to cover these surface integral conditions is fairly trivial. If we introduce a new set of parameters into $\overset{i}{\underset{r}{h}}{}^{0n}$ by writing

$$\overset{i}{\underset{r}{h}}{}^{0n} = \overset{i}{\underset{r}{h}}{}^{\prime 0n} + 4\sum_{i}\varepsilon^{nst}\overset{i}{\underset{r}{A}}{}_{s}\overset{i}{\underset{r}{\Psi}}_{,t}, \tag{4.14}$$

then $\overset{i}{\underset{r}{h}}{}^{0n}$ will be a solution of the field equations, (3.2), whenever $\overset{i}{\underset{r}{h}}{}^{\prime 0n}$ is. This can be shown by direct substitution in (3.2). If the second set of surface integrals are calculated, then (4.9), gives

$$\overset{i}{\underset{r}{\mathfrak{A}}}{}_{n} = 4\overset{i}{\underset{r-1}{A}}{}_{n,0} - 4\varepsilon_{nst}\left(\overset{i}{\underset{r-1}{D}}{}^{s}\overset{i}{\underset{r}{\dot{v}}}{}^{t} + \tfrac{1}{2}\overset{i}{\underset{r-1}{D}}{}^{s}\overset{i}{\underset{r}{\dot{v}}}{}^{t}\right) + \overset{i}{\underset{r}{\mathfrak{A}}}{}^{\prime}_{n} = 0, \tag{4.15}$$

where $\overset{i}{\underset{r}{\mathfrak{A}}}{}^{\prime}_{n}$ is independent of $\overset{i}{\underset{r-1}{A}}{}_{n}$. This gives a set of differential equations to be satisfied by the $\overset{i}{\underset{r}{A}}{}_{n}$ in each approximation. We shall not prove (4.15) here, since the calculations are trivial. All that we are interested in this section is the general method of the « new » approximation procedure.

(*) $\overset{i}{v}{}^{n}$ is the velocity of the i'th particle, $\overset{i}{v}{}^{n} = (d/dt)(\overset{i}{\xi}{}^{n})$. We shall denote differentiation with respect to the time coordinate, t or x^0, by a $(\dot{})$ over the parameter, *e.g.* $\dot{v} = (d/dt)(v)$.

(4.12), (4.13) and (4.15) give seven differential equations to be satisfied by the parameters of each particle in every approximation. Instead of there being only a finite number of differential equations for each particle, there is a set to be satisfied *in each approximation order*. EINSTEIN and INFELD gave the first clue to the interpretation of these equations by observing that if we define

$$\overset{i}{m} = \sum_r \overset{i}{m}_r , \qquad \overset{i}{D}{}^n = \sum_r \overset{i}{D}{}^n_r , \tag{4.16}$$

as the *physical* masses and dipole moments, then, by summing (4.12) and (4.13) over the approximation order index, r, we have

$$4\overset{i}{m}_{,0} = -\overset{i}{\mathfrak{C}}{}^{\prime 0} = -\sum_r \overset{i}{\mathfrak{C}}{}^{\prime n}_r(\overset{j}{m}_r, \overset{j}{D}{}^n_r, \ldots; s<r; \overset{i}{\xi}{}^n) , \tag{4.17,a}$$

$$4\overset{i}{\ddot{D}}{}^n - 4\overset{i}{m}\overset{i}{\dot{v}}{}^n = \overset{i}{\mathfrak{C}}{}^{\prime n} = \sum_r \overset{i}{\mathfrak{C}}{}^{\prime n}_r(\overset{j}{m}_s, \overset{j}{D}{}^t_s, \ldots; s<r; \overset{i}{\xi}{}^s) . \tag{4.17,b}$$

These were to be interpreted as the physical equations of mass and motion for each particle, and the approximation equations, (4.12) etc., were to be neglected, altogether. Similarly, we can sum (4.15) over the approximation order,

$$\overset{i}{A}_{n,0} - \varepsilon_{nst}\left(\overset{i}{D}{}^s_{,0}\overset{i}{v}{}^t + \tfrac{1}{2}\overset{i}{D}{}^s\overset{i}{\dot{v}}{}^t\right) = -\tfrac{1}{4}\overset{i}{\mathfrak{A}}{}'_n . \tag{4.17,c}$$

Before these equations can be interpreted as the *only* physical equations of motion, it is necessary to prove the following theorem,

THEOREM II-A. *Provided that the integrations are carried out consistently, the limiting field,*

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \sum_{r=1}^{\infty} \overset{i}{h}{}^{\mu\nu}_r ,$$

will be a function of the total masses, dipole moments and angular momentum vectors, $\overset{i}{m}$, $\overset{i}{D}{}^n$ *and* $\overset{i}{A}_n$, *rather than of their individual r'th order components. Furthermore, the right hand side of the equations of motion,* (4.17), *are also functions of these physical parameters,*

$$\overset{i}{\mathfrak{C}}{}^{\prime\mu}(\overset{j}{m}_s, \overset{j}{D}{}^n_s, \overset{j}{A}_{n\,s}, \ldots) = \overset{i}{\mathfrak{C}}{}^{\prime\mu}(\overset{j}{m}, \overset{j}{D}{}^n, \overset{j}{A}_n, \ldots) ,$$

and similarly for the other three surface integrals.

If this theorem were not true, then we should have to give some physical interpretation to the individual r'th order equations, (4.12) etc. In the history

of the EIH method, no published attempt has been made to prove this theorem. EINSTEIN and INFELD (1949) did attempt to prove that the $\overset{i}{\underset{r}{\mathfrak{G}}}{}^{n}$ are independent of the ${}_{r}D^{m}_{-2}$ and hoped that these surface integrals would be independent of all the dipole moments. However, their proof was incorrect, and these surface integrals contain the usual classical forces on a dipole moment moving in an external Newtonian field. This has been proved by MOFFAT and KERR ([5]), and presumably by numerous other people.

If this theorem is true then the r'th order components of the expanded particle parameters, $\overset{i}{\underset{r}{m}}$ etc., do not appear in the physical field, $g^{\mu\nu}$, and consequently it would appear pointless to expand these particle parameters. What we want is a method in which we do not have to expand the particle parameters nor calculate any non-physical functions such as the $\mathfrak{D}^{\mu}$ in the original EIH procedure of Section 3. We shall develop such a method in Section 6.

We shall not prove Theorem II-A here, since it follows as a trivial corollary of Theorem V, just as Theorem II does. Furthermore, once V is proved II and II-A are redundant, since in V we shall show how to avoid expanding the particle parameters or introducing the $\mathfrak{D}^{\mu}$.

5. – The non-singular field equations.

FOCK ([4]) and PAPAPETROU ([5]) proposed to solve the EIH approximation equations by introducing a *non-singular*, localized, source function, $\mathfrak{T}^{\mu\nu}$,

$$\overset{}{\underset{r}{\mathfrak{G}}}{}^{\mu\nu} \equiv \overset{}{\underset{r}{\Phi}}{}^{\mu\nu} + \overset{}{\underset{r}{\Lambda}}{}^{\mu\nu} = -8\pi \overset{}{\underset{r}{\mathfrak{T}}}{}^{\mu\nu} . \tag{5.1}$$

This was to be zero in the region V between the particles and non-singular inside. Unlike the $\mathfrak{T}^{\mu\nu}$ that EINSTEIN and INFELD introduced in 1940, the Fock source function was to represent the physical stress-energy tensor.

From equation (4.4), we see that (5.1) can be solved only if

$$\overset{}{\underset{r}{\Lambda}}{}^{\mu n}{}_{,n} + 8\pi \overset{}{\underset{r}{\mathfrak{T}}}{}^{\mu n}{}_{,n} = 0 . \tag{5.2}$$

Equation (5.2) is to be satisfied by a *localized* $\overset{}{\underset{r}{\mathfrak{T}}}{}^{\mu\nu}$. Let us suppose, as usual, that the particles are enclosed by a set of *convex* surfaces, S^{i}. From Section 3 we see that necessary and sufficient conditions for the existence of a solution of the field equations in V, where the stress-energy, tensor is zero, are that

$$\overset{}{\underset{r}{\Lambda}}{}^{\mu\nu}{}_{,n} = 0 , \quad \text{(in V)} , \tag{5.3}$$

and that the surface integral conditions be satisfied,

$$\overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu} = \frac{1}{2\pi}\int \overset{i}{\underset{r}{\Lambda}}{}^{\mu n}\,\mathrm{d}S_n = 0\,, \tag{5.4,a}$$

$$\overset{i}{\underset{r}{\mathfrak{A}}}_{m} = \varepsilon_{mst}\frac{1}{4\pi}\int \overset{i}{\underset{r}{\Lambda}}{}^{tu}\,\overset{i}{x}{}^{s}\,\mathrm{d}S_u = 0\,. \tag{5.4,b}$$

We shall show that these are also necessary and sufficient conditions for the existence of a localised solution of (5.2).

Let us consider the equations

$$\mathfrak{T}^{\mu n}{}_{,n} = \mathfrak{J}^{\mu}\,, \tag{5.5}$$

where $\mathfrak{J}^{\mu}$ is localized inside the S^i. We shall calculate the necessary and sufficient conditions for the existence of a localized $\mathfrak{T}^{\mu\nu}$ satisfying (5.5). If we integrate (5.5) over a three dimensional volume enclosing the i'th singularity, we see that

$$\int \overset{i}{\mathfrak{J}}{}^{\mu}\,\mathrm{d}^3x = 0\,. \tag{5.6,a}$$

Similarly, since

$$\varepsilon_{mst}(\mathfrak{T}^{tu}\overset{i}{x}{}^{s})_{,u} = \varepsilon_{mst}\overset{i}{x}{}^{s}\mathfrak{J}^{t}\,,$$

and the left hand side is a divergence,

$$\varepsilon_{mst}\int \overset{i}{x}{}^{s}\,\overset{i}{\mathfrak{J}}{}^{t}\,\mathrm{d}^3x = 0\,, \tag{5.6,b}$$

equations (5.6) are necessary conditions for there the existence of a *localized* solution of (5.5). Let us suppose that they are satisfied.

We may always write

$$\mathfrak{T}^{0n} = \varphi^0{}_{,n} + \varepsilon^{nst}\varphi_{s,t}\,. \tag{5.7}\ (^*)$$

This satisfies (5.5), for all φ_s, provided that

$$\begin{cases} \Delta^2\varphi^0 = \mathfrak{J}^0\,, \\ \varphi^0(r) = \dfrac{1}{4\pi}\displaystyle\int \frac{\mathfrak{J}^0(r')}{|r-r'|}\,\mathrm{d}^3x'\,. \end{cases} \tag{5.8}$$

(*) For simplicity, we shall only consider *one* particle localized within a sphere, S, whose centre is the origin of the coordinates.

φ_n is defined by

$$\varphi_n = \varepsilon_{nst} B_{s,t} , \tag{5.9}$$

where

$$B_s = \frac{1}{4\pi} \cdot \int \Im^0 x'^s \int_0^1 \frac{1}{|r - \alpha r'|} \, \mathrm{d}\alpha \, \mathrm{d}^3 x' , \tag{5.10}$$

outside the particle, and is continued into the interior so that it is differen tiable and non-singular. Since B_s is a harmonic function, we see that

$$\mathfrak{T}^{0n} = (\varphi^0 + B_{s,s})_{,n} ,$$

outside S. However, outside S we have

$$\begin{aligned} B_{s,s} = & \frac{1}{4\pi} \int \Im^0 \int_0^1 x'^s \frac{\partial}{\partial x^s} \left(\frac{1}{|r - \alpha r'|} \right) \mathrm{d}\alpha \, \mathrm{d}^3 x' \\ = & - \frac{1}{4\pi} \int \Im^0(r') \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha} \left(\frac{1}{|r - \alpha r'|} \right) \mathrm{d}\alpha \, \mathrm{d}^3 x' \\ = & - \frac{1}{4\pi} \int \frac{\Im^0(r')}{|r - r'|} \, \mathrm{d}^3 x' + \frac{1}{4\pi} \Psi \cdot \int \Im^0(r') \, \mathrm{d}^3 x' . \end{aligned}$$

From (5.6,a), we see that $\mathfrak{T}^{0n}$, as defined by (5.7) through (5.10) is zero outside S.

Similarly, we shall write $\mathfrak{T}^{mn}$ as

$$\begin{aligned} \mathfrak{T}^{mn} = & (\varrho^m - E^{m\underline{s}}{}_{,s} - F^{m:\underline{st}}{}_{,st})_{,n} + (\varrho^n - E^{n\underline{s}}{}_{,s} - F^{n:\underline{st}}{}_{,st})_{,m} - \\ & - \delta^{mn} (\varrho^u - E^{\underline{us}}{}_{,s} - F^{u:\underline{st}}{}_{,st})_{,u} + \Delta^2 (E^{m\underline{n}} + F^{m:\underline{ns}}{}_{,s} + F^{n:\underline{ms}}{}_{,s} - F^{s:\underline{mn}}{}_{,s}) , \end{aligned} \tag{5.11}$$

where $E^{m\underline{n}}$ and $F^{m:\underline{mn}}$ are symmetric in m and n. The right hand side of (5.11) will satisfy (5.5), provided that

$$\mathfrak{T}^{mn}{}_{,n} \equiv \varrho^m{}_{,ss} = \Im^m ,$$

i.e. provided that

$$\varrho^m = \frac{1}{4\pi} \int \frac{\Im^m}{|r - r'|} \, \mathrm{d}^3 x' . \tag{5.12}$$

If we expand $|r-r'|^{-1}$ about the origin, we have

$$\varrho^m = \sum_t \frac{(-)^t}{4\pi t!}\int \Im^m(r')\,x'^{(s)_t}\,\mathrm{d}^3x'\cdot\Psi_{,(s)_t} = E^{ms}{}_{,s} + F^{m;st}{}_{,st}\,, \qquad \text{(outside } S\text{)}, \tag{5.13}$$

where

$$\begin{cases} E^{ms} = -\dfrac{1}{4\pi}\displaystyle\int \Im^m(r')x'^s\,\mathrm{d}^3x'\cdot\Psi\,, \\ F^{m;uv} = \displaystyle\sum_t \frac{(-)^t}{4\pi(t+2)!}\int \Im^m(r')x'^u x'^v x'^{(s)_t}\,\mathrm{d}^3x'\cdot\Psi_{,(s)_t}\,, \end{cases} \tag{5.14}$$

outside S. $E^{\cdot\cdot}$ and $F^{\cdot\cdot}$ are continued into the interior of S so that they are non-singular and differentiable. (5.13) follows from (5.6,a), and E^{ms} is symmetric because of (5.6,b). From (5.14), E and F are harmonic functions outside S, and so, from (5.11) and (5.13), $\mathfrak{T}^{mn}$ is zero outside S. Consequently, we have calculated a non-singular, localized solution of (5.5).

Theorem III. Necessary and sufficient conditions for the existence of a localized solution of the equations, $\mathfrak{T}^{\mu n}{}_{,n} = \mathfrak{F}^{\mu}$, where $\mathfrak{F}^{\mu}$ is localized inside the surfaces S^i, are that the integral conditions, (5.6,a) and (5.6,b) should be satisfied.

Since $\overset{i}{\underset{r}{\Lambda}}{}^{\mu\nu}$ is non-singular, the surface integral conditions, (5.4), can be written as

$$\overset{i}{\underset{r}{\mathfrak{S}}}{}^{\mu} = \frac{1}{2\pi}\int \overset{i}{\underset{r}{\Lambda}}{}^{\mu n}{}_{,n}\,\mathrm{d}^3x = 0\,, \tag{5.15,a}$$

$$\overset{i}{\underset{r}{\mathfrak{A}}}{}_m = \varepsilon_{mst}\frac{1}{4\pi}\int \overset{i}{x}{}^s \overset{i}{\underset{r}{\Lambda}}{}^{tu}{}_{,u}\,\mathrm{d}^3x = 0\,, \tag{5.15,b}$$

where the integration is over the volume enclosed by S^i. From Theorem III, we see that (5.4) is sufficient to guarantee the existence of a localized solution of (5.2). (5.2) is then sufficient to guarantee that (5.1) can be solved.

We have seen that in order to solve (5.1) we have to satisfy the surface integral conditions of (5.4). Just as in the EIH methods, this gives seven conditions on the lower order fields for each particle and in every approximation. Fock [4] and Papapetrou [5] showed how this might be avoided.

From (3.1) through (3.3), we see that if we introduce the De Donder coordinate conditions,

$$\mathfrak{g}^{\mu\nu}{}_{,\nu} = h^{\mu n}{}_{,n} + h^{\mu 0}{}_{,0} = 0\,, \tag{5.16}$$

then the field equations reduce to

$$\underset{r}{h}{}^{\mu\nu}{}_{,ss} = 2\overset{*}{\underset{r}{\Lambda}}{}^{\mu\nu} + 16\pi\underset{r}{\mathfrak{T}}{}^{\mu\nu}\,, \tag{5.17}$$

where

$$\text{(5.18)}\qquad \begin{cases} \overset{*}{\underset{r}{\Lambda}}{}^{00} = \underset{r}{\Lambda}{}^{00} + \frac{1}{2}\,\underset{r-2}{h}{}^{00}{}_{,00} \\ \overset{*}{\underset{r}{\Lambda}}{}^{0n} = \underset{r}{\Lambda}{}^{0n} - \frac{1}{2}\,\underset{r-1}{h}{}^{00}{}_{,0n}\,, \\ \overset{*}{\underset{r}{\Lambda}}{}^{mn} = \underset{r}{\Lambda}{}^{mn} - \frac{1}{2}\left(\underset{r-1}{h}{}^{m0}{}_{,0n} + \underset{r-1}{h}{}^{n0}{}_{,0m} + \delta^{mn}\,\underset{r-2}{h}{}^{00}{}_{,00}\right). \end{cases}$$

Equations (5.17) can be solved for *arbitrary* $\underset{r}{\mathfrak{T}}{}^{\mu\nu}$, since they are Poisson equations. Once the field has been calculated to the r'th approximation, Fock and Papapetrou proposed to impose the conditions

$$\text{(5.19)}\qquad \underset{(r-1)}{\mathfrak{T}}{}^{\mu 0}{}_{,0} + \underset{(r)}{\mathfrak{T}}{}^{\mu n}{}_{,n} + \sum_s \underset{s}{\Gamma}{}^{\mu}_{\alpha\beta}\,\underset{(r-s)}{\mathfrak{T}}{}^{\alpha\beta} = 0\,,$$

to the r'th approximation. They did not attempt to satisfy these equations exactly, since then they would have had to satisfy the integral conditions in every approximation, but proposed that they should be satisfied to the r'th approximation. (*)

We shall now show how the Fock equations, (5.17), (5.16) and (5.19) can be satisfied without introducing the integrability equations in each order. We shall relax the divergence relations on the r'th order source functions and write them as

$$\text{(5.20)}\qquad \underset{r}{\mathfrak{T}}{}^{\mu n}{}_{,n} = -\,\underset{r-1}{\mathfrak{T}}{}^{\mu 0}{}_{,0} - \sum_s \underset{s}{\Gamma}{}^{\mu}_{\alpha\beta}\,\underset{r-s}{\mathfrak{T}}{}^{\alpha\beta} - \sum_i \left(\overset{i}{\underset{r}{k}}{}^{\mu} + \overset{i}{\underset{r}{k}}{}^{\mu n}\,\overset{i}{x}{}^{n}\right) f(\overset{i}{r})\,,$$

where $\overset{i}{\underset{r}{k}}{}^{\mu\nu}$ is an antisymmetric tensor whose $(\mu 0)$ components are zero, *i.e.* $k^{\mu 0} = 0$.

We shall suppose that the surfaces, S^i, are spherically symmetric, and that the Einstein field equations are to be satisfied in the region, V, outside the S^i. $f(\overset{i}{r})$ is a spherically symmetric function, zero outside S^i, and satisfying

$$\text{(5.21)}\qquad \begin{cases} \int f(r)\,\mathrm{d}^3x = 1\,, \\ \int x^s x^t f(r)\,\mathrm{d}^3x = \delta^{st}\,. \end{cases}$$

These are just normalization conditions. From Theorem III, (5.20) can be solved for a $\underset{r}{\mathfrak{T}}{}^{\mu\nu}$, localized in the S^i, provided that

$$\text{(5.22)}\qquad \overset{i}{\underset{r}{\mathfrak{C}}}{}^{\mu} = -\,4\int_{(i)} \left(\underset{r-1}{\mathfrak{T}}{}^{\mu 0}{}_{,0} + \sum_s \underset{s}{\Gamma}{}^{\mu}_{\alpha\beta}\,\underset{r-s}{\mathfrak{T}}{}^{\alpha\beta} + \overset{i}{\underset{r}{k}}{}^{\mu} f(\overset{i}{r}) + \overset{i}{\underset{r}{k}}{}^{\mu n}\,\overset{i}{x}{}^{n} f(\overset{i}{r})\right) \mathrm{d}^3x = 0\,,$$

$$\text{(5.23)}\qquad \overset{i}{\underset{r}{\mathfrak{A}}}{}_{m} = 2\varepsilon_{mst}\int_{(i)} \overset{i}{x}{}^{s}\left(\underset{r-1}{\mathfrak{T}}{}^{t0}{}_{,0} + \sum_s \underset{s}{\Gamma}{}^{t}_{\alpha\beta}\,\underset{r-s}{\mathfrak{T}}{}^{\alpha\beta} + \ldots\right) \mathrm{d}^3x = 0\,.$$

(*) *I.e.*, the L.H.S. of (5.19) was to be a function of the $(r+1)$'th, or higher order.

These can be written as

$$(5{\cdot}24,a) \qquad \overset{i}{k}{}^{\mu}_{r} = -\int^{i} (\overset{\mathfrak{T}}{}_{r-1}{}^{\mu 0}{}_{,0} + \sum_{s} \Gamma^{\mu}_{s\,\alpha\beta}\, \overset{\mathfrak{T}}{}_{r-s}{}^{\alpha\beta})\, \mathrm{d}^3x\,,$$

$$(5.24,b) \qquad \overset{i}{k}{}^{mn}_{r} = \frac{1}{2}\,\varepsilon^{mnr}\varepsilon_{rst}\int^{i} \overset{i}{x}{}^{s} (\overset{\mathfrak{T}}{}_{r-1}{}^{t0}{}_{,0} + \sum_{s} \Gamma^{\mu}_{s\,\alpha\beta}\, \overset{\mathfrak{T}}{}_{r-s}{}^{\alpha\beta})\, \mathrm{d}^3x\,.$$

We can solve (5.20), provided that the k's are given by (5.24).

We shall define an O_n-function, $\underset{(n)}{P}$ say, as a function of the following form,

$$(5.25) \qquad \underset{(n)}{P} = \sum_{r=1}^{r=n} \underset{r}{P} = \sum_{i,s,u} \overset{i}{k}{}^{A}_{(s)}\, {}_{\leftarrow u \rightarrow}^{,00..0}\, \overset{i}{P}{}_{A}(n-s-u, s)\,,$$

where the $\overset{i}{k}{}^{A}_{s}$ are $(\overset{i}{k}{}^{\mu}_{s}, \overset{i}{k}{}^{mn}_{s})$, and as usual

$$\overset{i}{k}{}^{A}_{(s)} = \sum_{t=1}^{t=s} \overset{i}{k}{}^{A}_{t}\,.$$

$\overset{i}{P}{}_{r\,A}(n, s)$ is of the r'th approximation order. Let us suppose that $\overset{i}{k}{}^{A}_{(\infty)}$, *i.e.* $\overset{i}{k}{}^{A}$, is zero,

$$\overset{i}{k}{}^{A} = \sum_{s=1}^{\infty} \overset{i}{k}{}^{A}_{s} = 0\,.$$

Then

$$\overset{i}{k}{}^{A}_{(s)}\, {}_{\leftarrow u \rightarrow}^{,0..00} = -\sum_{r=1}^{\infty} \overset{i}{k}{}^{A}_{s+r}\, {}_{\leftarrow u \rightarrow}^{,0..00}\,,$$

which is of the $(s+u+1)$'th, or higher, order. Consequently, from (5.25), $\underset{(n)}{P}$ will be of the $(n+1)$'th, or higher order. This shown that all O_n-functions are zero to the n'th approximation whenever the $\overset{i}{k}{}^{A}_{(\infty)}$ are zero.

Let us suppose as our induction hypothesis that (5.24), (5.20) and (5.17) are satisfied for all $r < n$, and that

$$(5.26) \qquad \underset{(n-1)}{h}{}^{\mu s}{}_{,s} + \underset{(n-2)}{h}{}^{\mu 0}{}_{,0} = O_{n-1}\text{-function}\,.$$

From the definition of an O_n-function, (5.25), we see that if $\underset{(n)}{P}$ is an O_n-function then $\underset{(r)}{P}$ is an O_r-function, for all $r \leqslant n$. Consequently, (5.26) implies that

$$(5.27) \qquad \underset{(r)}{z}{}^{\mu} \equiv \underset{(r)}{h}{}^{\mu s}{}_{,s} + \underset{(r-1)}{h}{}^{\mu 0}{}_{,0} = O_r\text{-function}\,, \qquad r < n.$$

The defines $\underset{s}{z}{}^{\mu}$. We shall now show how the n'th order field equations may be solved so that $\underset{(n)}{z}{}^{\mu}$ is an O_n-function.

Let $\overset{r}{\mathfrak{T}}{}^{\mu\nu}$ be any localized solution of (5.20), where the $\overset{i}{k}{}^{A}$ are given by (5.24). Also, let $\overset{n}{\bar{h}}{}^{\mu\nu}$ be a solution of the Poisson equations, (5.17). If we take the divergence of the field equations, (5.17), we obtain

$$(5.28)\qquad \begin{cases} \overset{n}{\bar{h}}{}^{0t}{}_{,tss} = 2\left(\overset{*}{\underset{n}{\Delta}}{}^{0t}{}_{,t} + 8\pi\overset{n}{\mathfrak{T}}{}^{0t}{}_{,t}\right) = -\,\underset{n-1}{h}{}^{00}{}_{,0tt} + 2\left(\underset{n}{\Delta}{}^{0t} + 8\pi\overset{n}{\mathfrak{T}}{}^{0t}\right)_{,t}, \\ \overset{n}{\bar{h}}{}^{mt}{}_{,tss} = 2\left(\overset{*}{\underset{n}{\Delta}}{}^{mt}{}_{,t} + 8\pi\overset{n}{\mathfrak{T}}{}^{mt}{}_{,t}\right) = -\,\underset{n-1}{h}{}^{m0}{}_{,0ss} - \\ \qquad\qquad - \left(\underset{n-1}{h}{}^{0s}{}_{,s} + \underset{n-2}{h}{}^{00}{}_{,0}\right)_{,0m} + 2\left(\underset{n}{\Delta}{}^{mt}{}_{,t} + 8\pi\overset{n}{\mathfrak{T}}{}^{mt}{}_{,t}\right). \end{cases}$$

Also, the Fock-Papapetrou equations may be written as

$$(5.29)\qquad \begin{cases} 2\underset{r}{\mathfrak{G}}{}^{00} = -\,\underset{r}{h}{}^{00}{}_{,ss} + \underset{r}{h}{}^{st}{}_{,st} + 2\underset{r}{\Delta}{}^{00} \\ \qquad = -\,\underset{r}{h}{}^{00}{}_{,ss} + 2\overset{*}{\underset{r}{\Delta}}{}^{00} + \left(\underset{r}{h}{}^{ts}{}_{,s} + \underset{r-1}{h}{}^{t0}{}_{,0}\right)_{,t} - \left(\underset{r-1}{h}{}^{0t}{}_{,t} + \underset{r-2}{h}{}^{00}{}_{,0}\right)_{,0} \\ \qquad = -\,16\pi\underset{r}{\mathfrak{T}}{}^{00} + \underset{r}{z}{}^{s}{}_{,s} - \underset{r-1}{z}{}^{0}{}_{,0}, \\ 2\underset{r}{\mathfrak{G}}{}^{0n} = -\,16\pi\underset{r}{\mathfrak{T}}{}^{0n} + \underset{r}{z}{}^{0}{}_{,n}, \\ 2\underset{r}{\mathfrak{G}}{}^{mn} = -\,16\pi\underset{r}{\mathfrak{T}}{}^{mn} + \underset{r}{z}{}^{m}{}_{,n} + \underset{r}{z}{}^{n}{}_{,m} - \delta^{mn}\left(\underset{r}{z}{}^{s}{}_{,s} - \underset{r}{z}{}^{0}{}_{,0}\right), \end{cases}$$

that is

$$(5.30)\qquad \underset{(r)}{\mathfrak{G}}{}^{\mu\nu} = -\,8\pi\underset{(r)}{\mathfrak{T}}{}^{\mu\nu} + O_r\text{-function}, \qquad r < n.$$

If we sum equation (4.5), which is derived from the Bianchi identities, over the approximation order index, r, we obtain

$$(5.31)\qquad \underset{(n)}{\Delta}{}^{\mu t}{}_{,t} = -\,\underset{(n-1)}{\mathfrak{G}}{}^{\mu 0}{}_{,0} - \sum_{s} \underset{s}{\Gamma}{}^{\mu}{}_{\alpha\beta}\,\underset{(n-s)}{\mathfrak{G}}{}^{\alpha\beta}.$$

Since an O_{n-s}-function multiplied by a function of the s'th approximation order gives an O_n-function, (5.31) may be written as

$$(5.32)\qquad \underset{(n)}{\Delta}{}^{\mu t}{}_{,t} = 8\pi\left(\underset{(n-1)}{\mathfrak{T}}{}^{\mu 0}{}_{,0} + \sum_{s} \underset{s}{\Gamma}{}^{\mu}{}_{\alpha\beta}\,\underset{(n-s)}{\mathfrak{T}}{}^{\alpha\beta}\right) + O_n\text{-function}.$$

Consequently, from (5.18), (5.20) and this equation,

$$(5.32')\qquad \overset{*}{\underset{(n)}{\Delta}}{}^{\mu t}{}_{,t} + 8\pi\underset{(n)}{\mathfrak{T}}{}^{\mu t}{}_{,t} = -\,\tfrac{1}{2}\,\underset{(n-1)}{h}{}^{\mu 0}{}_{,0ss} + O_n\text{-function}.$$

These equation are not identities, but are satisfied whenever the Fock-Papapetrou field equations, (5.17), and equations (5.20) and (5.26) are satisfied.

From (5.28) and (5.32),

$$(5.33)\qquad \Delta^{2}\left(\overset{n}{\bar{h}}{}^{\mu s}{}_{,s} + \underset{(n-1)}{h}{}^{\mu 0}{}_{,0}\right) = O_n\text{-function},$$

where

$$\bar{h}^{\mu\nu}_{(n)} = \bar{h}^{\mu\nu}_{n} + h^{\mu\nu}_{(n-1)} .$$

Since the field is non-singular, (5.33) implies that

$$\bar{h}^{\mu t}_{(n)\,,t} + h^{\mu 0}_{(n-1)\,,0} = \varphi^{\mu}_{(n)} + O_n\text{-function} , \tag{5.34}$$

where $\varphi^{\mu}_{(n)}$ is a non-singular harmonic function. However, from the definition of an O_n-function, we see that when we omit the terms of the n'th order we obtain an O_{n-1}-function, and so

$$h^{\mu t}_{(n-1)\,,t} + h^{\mu 0}_{(n-2)\,,0} = \varphi^{\mu}_{(n-1)} + O_{n-1}\text{-function} .$$

From this equation and our induction hypothesis, (5.26), we have

$$\varphi^{\mu}_{(n-1)} = O_{n-1}\text{-function} = \sum_{i,A,s,u} {}^{i}_{(s}k^{A}_{-1)}\,{}_{,0\ldots00}^{\leftarrow u\rightarrow}\ {}^{i}_{n-s-n}\varphi_{A}(n-1,\ s) ,$$

where, since the ${}^{i}_{(t}k^{A}_{)}$ are just coefficients, and since $q^{\mu}_{(n)}$ is a harmonic function, the ${}^{i}_{r}\varphi_{A}(n-1,\ s)$ are harmonic functions of the r'th order. Therefore,

$$\varphi^{\mu}_{(n)} = \varphi^{\mu}_{(n-1)} + \varphi^{\mu}_{n} = \sum_{i,A,s,u} {}^{i}_{(s}k^{A}_{)}\,{}_{,0\ldots00}^{\leftarrow u\rightarrow}\ {}^{i}_{n-s-u}\varphi_{A}(n-1,\ s) + {}^{*}_{n}\varphi^{\mu} ,$$

where ${}^{*}_{n}\varphi^{\mu}$ is a non-singular harmonic function of the n'th order,

$$ {}^{*}_{n}\varphi^{\mu} = \varphi^{\mu}_{n} - \sum_{i,A,s,u} {}^{i}_{s}k^{A}_{}\,{}_{,0\ldots00}^{\leftarrow u\rightarrow}\ {}^{i}_{n-s-u}\varphi_{A}(n-1,\ s) ,$$

and where it should be noticed that the summation is linear in the ${}^{i}_{s}k^{A}_{}$, rather than their partial sums, ${}^{i}_{(s}k^{A}_{)}$. Consequently,

$$\bar{z}^{\mu}_{(n)} = \bar{h}^{\mu t}_{(n)\,,t} + h^{\mu 0}_{(n)\,,0} = {}^{*}_{n}\varphi^{\mu} + O_n\text{-function} . \tag{5.35}$$

By adding a non-singular harmonic function, $\chi^{\mu\nu}_{n}$, onto $h^{\mu\nu}_{n}$, as in (3.15) through (3.19), we obtain an $h^{\mu\nu}_{n}$ which satisfies

$$z^{\mu}_{(n)} = h^{\mu n}_{(n)\,,n} + h^{\mu 0}_{(n-1)\,,0} = O_n\text{-function} . \tag{5.36}$$

By induction, this shows that we may always solve the field equations in such a way that (5.36) is satisfied, since it is trivially true for $n = 0$.

THEOREM V. *The Fock-Papapetrou field equations,* (5.17), *can be solved in such a way that*

$$\text{(V,}a\text{)} \qquad \mathfrak{T}_n^{\mu s}{}_{,s} + \mathfrak{T}_{n-1}^{\mu 0}{}_{,0} + \sum_s \Gamma^{\mu}_{s\,\alpha\beta}\mathfrak{T}_{n-s}^{\alpha\beta} = O_n\text{-function}\,,$$

$$\text{(V,}b\text{)} \qquad \underset{(n)}{h}{}^{\mu s}{}_{,s} + \underset{(n-1)}{h}{}^{\mu 0}{}_{,0} = O_n\text{-function}\,,$$

where the $\overset{i}{k}{}_s^{\mu}$ *and* $\overset{i}{k}{}_s^{mn}$ *are defined by* (5.24). *With this definition of the* $\overset{i}{k}{}_s^{4}$, $\mathfrak{T}_r^{\mu\nu}$ *will be a localized source function. If we equate the* $\overset{i}{k}{}^{A}$ *to zero,*

$$\text{(V,}c\text{)} \qquad \overset{i}{k}{}^{A} = \sum_{s=1}^{\infty} \overset{i}{k}{}_s^{A} = 0\,,$$

then all O_n-functions will be zero to the n'th approximation, and so the left hand side of (V,a) and (V,b) will be zero to the n'th order. Also, the Einstein field equations will be satisfied to the n'th approximation.

To prove the last part of this theorem, we observe that (5.29) and (V,b) imply that

$$\mathfrak{G}_{(n)}^{\mu\nu} = -8\pi\mathfrak{T}_{(n)}^{\mu\nu} + O_n\text{-function}\,,$$

for all n. Consequently,

$$\begin{aligned}\mathfrak{G}^{\mu\nu}(\underset{(n)}{h}{}^{\alpha\beta}) &= -8\pi\mathfrak{T}_{(n)}^{\mu\nu} + O_n\text{-function} + \sum_{s>0}\mathfrak{G}_{n+s}^{\mu\nu}(\underset{1}{h}{}^{\alpha\beta}, \ldots, \underset{n}{h}{}^{\alpha\beta}) \\ &= -8\pi\mathfrak{T}_{(n)}^{\mu\nu} + \text{function of the } (n+1)\text{'th order}\,.\end{aligned}$$

This shows that $\underset{(n)}{h}{}^{\mu\nu}$ will satisfy the field equations to the n'th order whenever the equations of motion, (V,c), are satisfied to the same order. It is not necessary to equate the separate $\overset{i}{k}{}_s^{A}$ to zero.

This was the first method to give the equations of motion without it being necessary to introduce seven parameters into each approximation, or to introduce the non-physical $\mathfrak{D}''$ of the 1940 EIH procedure. The reason for this was that the Einstein equations were reduced to a set of Poisson equations, plus a set of linear De Donder conditions. Instead of demanding that the coordinate conditions should be satisfied exactly in every approximation, FOCK and PAPAPETROU relaxed this slightly so that $\underset{(n)}{z}{}^{\mu}$ was what we have called an O_n-function, that is a function which is zero *to the* n'th *approximation* whenever the equations of motion are satisfied to the same order.

We should observe that FOCK and PAPAPETROU never suggested this relaxation of the coordinate conditions in their papers, nor did they prove that their method was consistent, as in Theorem V. They assumed that if (V,a) is satisfied then (V,b) is also.

It has been usual in the EIH expansion to assume that $\overset{i}{h}^{\mu\nu}$ is zero so that the expansion is started with terms of the second order. This does not result in any loss of generality but does ensure that one obtains the Newtonian equations of motion in the lowest approximation. If this is done, then the integrands of the $\overset{i}{k}^{A}_{(n)}$ are independent of the field to the $(n-1)$'th and n'th orders, from (5.24). It is only necessary to know $\mathfrak{T}^{\mu 0}$ to the $(n-1)$'th order, $\mathfrak{T}^{\mu m}$ to the $(n-2)$'th order, and the field to the $(n-2)$'th order to obtain the equations of motion, mass and angular momentum to the n'th approximation. It would appear from this that the Fock-Papapetrou method is superior to the EIH procedures, since in the latter it is necessary to know the field to the $(n-1)$'th order before the n'th order surface integrals, and so the equations of motion to the same order, can be calculated. However, we do not believe that there is any significant difference in the amount of computations involved in the two methods, since one has to calculate the source function in the Fock-Papapetrou procedure. Both the singularity approach and the hydrodynamical approach to calculating the equations of motion serve to illuminate the problem.

6. – The reduced field equations.

In this section we shall derive a set of approximation equations which are related to the EIH field equations. We shall show that these equations can be solved without assuming any specific coordinate conditions, without expanding the particle parameters, and without introducing a source function, whether localized or not.

From the Appendix, we see that the Einstein tensor density may be written as

$$\mathfrak{G}^{\mu\nu} = -\tfrac{1}{2}h^{\mu\nu}{}_{,ss} + \acute{\Lambda}^{\mu\nu} + \tfrac{1}{2}(g^{\nu\alpha}\mathfrak{g}^{\mu s}{}_{,s\alpha} + g^{\mu\alpha}\mathfrak{g}^{\nu s}{}_{,s\alpha} - g^{\mu\nu}\mathfrak{g}^{\alpha s}{}_{,s\alpha}) - \\ - \tfrac{1}{2}(g^{\mu\nu}{}_{,\alpha} + g^{\mu\nu}\log(-g)_{,\alpha})\mathfrak{g}^{\alpha s}{}_{,s}\,, \tag{6.1}$$

where $\acute{\Lambda}^{\mu\nu}$ is defined in the Appendix and has no terms linear in the spatial derivatives of the $h^{\alpha\beta}$. We shall *define* a^{μ} by

$$a^{\mu} = \mathfrak{g}^{\mu s}{}_{,s}\,. \tag{6.2}$$

Since it is an arbitrary unspecified function of position, this is not a co-ordinate condition. We shall take as our field equations,

$$\acute{\mathfrak{G}}^{\mu\nu} \equiv -\tfrac{1}{2}h^{\mu\nu}{}_{,s} + \acute{\Xi}^{\mu\nu} \equiv -\tfrac{1}{2}h^{\mu\nu}{}_{,ss} + \acute{\Lambda}^{\mu\nu} + \\ + \tfrac{1}{2}(g^{\nu\alpha}a^{\mu}{}_{,\alpha} + g^{\mu\alpha}a^{\nu}{}_{,\alpha} - g^{\mu\nu}a^{\alpha}{}_{,\alpha}) - \tfrac{1}{2}(g^{\mu\nu}{}_{,\alpha} + g^{\mu\nu}\cdot\log(-g)_{,\alpha})a^{\alpha} = 0\,. \tag{6.3}$$

These are completely equivalent to the Einstein equations, provided that (6.2) is satisfied. If we introduce the quasi-static approximation into (6.3), we obtain the r'th field equations,

$$-\tfrac{1}{2}\underset{r}{h}^{\mu\nu}{}_{,ss} + \underset{r}{\Xi}^{\mu\nu} = 0\,, \tag{6.4}$$

where $\underset{r}{\Xi}^{\mu\nu}$ is a function of the $\underset{s}{a}^{\alpha}$, $s < r$, and of the lower order fields, $\underset{s}{h}^{\alpha\beta}$, $s < r$. Since these are just Poisson equations, they can always be solved.

If we define z^{μ} by

$$z^{\mu} = \mathfrak{g}^{\mu s}{}_{,s} - a^{\mu}\,, \tag{6.5}$$

then the Einstein tensor may be written as

$$\mathfrak{G}^{\mu\nu} \equiv \overset{\prime}{\mathfrak{G}}{}^{\mu\nu} + \tfrac{1}{2}(g^{\nu\alpha}z^{\mu}{}_{,\alpha} + g^{\mu\alpha}z^{\nu}{}_{,\alpha} - g^{\mu\nu}z^{\alpha}{}_{,\alpha}) - \tfrac{1}{2}(g^{\mu\nu}{}_{,\alpha} + g^{\mu\nu}\log(-g)_{,\alpha})z^{\alpha}\,. \tag{6.6}$$

From the Bianchi identities, (4.2), we have

$$\begin{aligned} -\tfrac{1}{2}\underset{(r)}{h}^{\mu n}{}_{,ssn} + \tfrac{1}{2}\sum_{s}\Big[\underset{s}{g}^{n\alpha}\underset{(r-s)}{z}^{\mu}{}_{,\alpha} - \underset{s}{g}^{\mu\alpha}\underset{(r-s)}{z}^{n}{}_{,\alpha} - \underset{s}{g}^{\mu n}\underset{(r-s)}{z}^{\alpha}{}_{,\alpha} - \\ - \big(\underbrace{g^{\mu n}{}_{,\alpha}}_{s} + \underbrace{g^{\mu n}\log(-g)_{,\alpha}}_{s}\big)\underset{(r-s)}{z}^{\alpha}\Big]_{,n} + \underset{(r)}{\Xi}^{\mu n}{}_{,n} + \underset{(r-1)}{\mathfrak{G}}^{\mu 0}{}_{,0} + \sum_{s}\underset{s}{\Gamma}^{\mu}_{\alpha\beta}\underset{(r-s)}{\mathfrak{G}}^{\alpha\beta} \equiv 0\,. \end{aligned} \tag{6.7}$$

This may be written as

$$\begin{aligned} \underset{(n)}{\Xi}^{\mu s}{}_{,s} - \tfrac{1}{2}\underset{(n)}{a}^{\mu}{}_{,ss} \equiv -\underset{(n-1)}{\mathfrak{G}}^{\mu 0}{}_{,0} - \sum_{s}\underset{s}{\Gamma}^{\mu}_{\alpha\beta}\underset{(n-s)}{\mathfrak{G}}^{\alpha\beta} + \tfrac{1}{2}\underset{(n-1)}{z}^{\lambda}{}_{,0s}\eta^{\mu s} - \tfrac{1}{2}\delta^{\mu}_{0}\underset{(n-1)}{z}^{s}{}_{,s0} + \\ + \tfrac{1}{2}\sum_{s>0}\Big[\underset{s}{h}^{\mu r}\underset{(n-s)}{z}^{\lambda}{}_{,\lambda} - \underset{s}{h}^{\mu\lambda}\underset{(n-1)}{z}^{r}{}_{,\alpha} - \underset{s}{h}^{r\alpha}\underset{(n-s)}{z}^{\mu}{}_{,\alpha} + \big(\underset{s}{h}^{\mu r}{}_{,\alpha} + \underbrace{g^{\mu r}\log(-g)_{,\alpha}}_{s}\big)\underset{(n-s)}{z}^{\lambda}{}_{,r}\Big]\,, \end{aligned} \tag{6.8}$$

where we understand $\underset{s}{z}^{\mu}{}_{,\lambda}$ to be $\underset{s}{z}^{\mu}{}_{,n}$ and ${}_{s}z^{\mu}_{1,0}$, for $\mu \neq 0$ and 0, respectively. Also,

$$\begin{aligned} \underset{(n)}{\mathfrak{G}}^{\mu\nu} \equiv \underset{(n)}{\overset{\prime}{\mathfrak{G}}}{}^{\mu\nu} + \tfrac{1}{2}\sum_{s}\Big[\underset{s}{g}^{\nu\alpha}\underset{(n-s)}{z}^{\mu}{}_{,\alpha} + \underset{s}{g}^{\mu\alpha}\underset{(n-s)}{z}^{\nu}{}_{,\alpha} - \underset{s}{g}^{\mu\nu}\underset{(n-s)}{z}^{\alpha}{}_{,\alpha} - \\ - \big(\underset{s}{g}^{\mu\nu}{}_{,\alpha} + \underbrace{g^{\mu\nu}\log(-g)_{,\alpha}}_{s}\big)\underset{(n-s)}{z}^{\alpha}\Big]. \end{aligned} \tag{6.9}$$

From the previous sections, we know that if we tried to satisfy (6.2) and (6.4) *exactly* in every approximation then we should have to satisfy seven surface integral conditions, for each particle, *in every approximation*. Consequently, we shall follow the method of Section 5, and instead of trying to satisfy $\underset{r}{z}^{\mu} = 0$, we shall relax this slightly to

$$\underset{(r)}{z}^{\mu} \equiv \underset{(r)}{h}^{\mu n}{}_{,n} - \underset{(r)}{a}^{\mu} = 0\,, \qquad \textit{to the r'th approximation.} \tag{6.10}$$

As in the previous section, we shall define an O_n-function by (5.25), where the $\overset{i}{k}_s^1(=\overset{i}{k}_s^{\mu}, \overset{i}{k}_s^{mn})$ will be defined later by induction. Furthermore, we shall define an $\overset{*}{O}_n$-function as an O_n-function in which the coefficient of $\overset{i}{k}_{(n)}^1$ is zero. Such a function will be zero to the n'th order whenever the equations of motion, $\overset{i}{k}_{(\infty)}^1 = 0$, are satisfied *to the* $(n-1)$*'th approximation.* An O_n-function is zero to the n'th order whenever the equations of motion are satisfied *to the* n*'th approximation.*

We shall now suppose as our induction hypothesis that the field equations, (6.4), have been satisfied in all orders up to the $(n-1)$'th,

$$\text{(H,1)} \qquad \overset{i}{\mathfrak{G}}_s^{\mu\nu} = 0\,, \qquad s < n,$$

and that $\overset{z}{}_{(n-1)}^{\mu}$ is an O_{n-1}-function of the following type,

$$\text{(H,2)} \qquad \overset{z}{}_{(n-1)}^{\mu} \equiv \underset{(n-1)}{\mathfrak{g}}^{\mu s}{}_{,s} - \underset{(n-1)}{a}^{\mu} = \sum_i \left(\overset{i}{k}_{(n-1)}^{\mu} \overset{i}{\Psi} + \overset{i}{k}_{(n-1)}^{\mu s} \overset{i}{\Psi}_{,s} \right) + \overset{*}{O}_{n-1}\text{-function}\,,$$

where, as usual, $\overset{i}{k}_s^{\mu 0} = 0$. The form of the right hand side of this equation is similar to that of (3.21). Later we shall see why this is so.

Let $\overset{}{\underset{n}{\bar{h}}}^{\mu\nu}$ be a particular solution of the reduced field equations, (6.4), in the n'th order, and let $\overset{}{\underset{(n)}{\bar{h}}}^{\mu\nu} = \underset{(n-1)}{h}^{\mu\nu} + \overset{}{\underset{n}{\bar{h}}}^{\mu\nu}$. Just as in the last chapter, see (5.26) and (5.27), (H,2) implies that

$$\text{(H,2}') \qquad \overset{z}{}_{(r)}^{\mu} = \sum_i \left(\overset{i}{k}_{(n)}^{\mu} \overset{i}{\Psi} + \overset{i}{k}_{(n)}^{\mu s} \overset{i}{\Psi}_{,s} \right) + \overset{*}{O}_r\text{-function}\,, \qquad r < n.$$

Also, the product of an O_r-function by a function of the s'th approximation order ($s > 0$) is an $\overset{*}{O}_{r+s}$-function, and the derivative of an O_r-function with respect to x^0 is an $\overset{*}{O}_{r+1}$-function. Consequently, from (6.9), (H,1) and (H,2'), we see that

$$(6.11) \qquad \overset{i}{\underset{(n)}{\mathfrak{G}}}^{\mu\nu} = O_s\text{-function}\,, \qquad s < n,$$

and so, from (6.8),

$$(6.12) \qquad \underset{(n)}{\mathfrak{g}}^{\mu s}{}_{,s} - \tfrac{1}{2} \underset{(n)}{a}^{\mu}{}_{,ss} = \overset{*}{O}_n\text{-function}\,.$$

Since $\underset{(n)}{h}^{\mu\nu}$ is a solution of the field equations, (6.4), summed over the approximation order index, r,

$$\overset{}{\underset{(n)}{\bar{h}}}^{\mu\nu}{}_{,ss} = 2 \underset{(n)}{\Xi}^{\mu\nu}\,,$$

we have, taking the spatial divergence of this equation, and using (6.12),

$$(6.13) \qquad \Delta^2 \left(\overset{}{\underset{(n)}{\bar{h}}}^{\mu s}{}_{,s} - \underset{(n)}{a}^{\mu} \right) = \overset{*}{O}_n\text{-function}\,.$$

Since the $\overset{i}{h}{}^{\lambda}_{(s)}$ in the definition of an $\overset{*}{O}_n$-function are only multiplying factors, (6.13) can be integrated to give

$$\overline{h}^{\mu s}_{(n)\,,s} - a^{\mu}_{(n)} = \text{harmonic function} + \overset{*}{O}_n\text{-function},$$

and hence

$$\overline{h}^{\mu s}_{(n)\,,s} - a^{\mu}_{(n)} = \varphi^{\mu}_{(n)} + \sum_{i,t} \overset{i}{A}{}^{\mu:(s)_t}_{(n)} \overset{i}{\Psi}_{,(s)_t} + \overset{*}{O}_n\text{-function}, \tag{6.14}$$

where $q^{\mu}_{(n)}$ is a non-singular harmonic function, and the $\overset{i}{A}{}^{\mu:(s)_t}_{(n)}$ are certain parameters which are completely symmetric in the s_i. As in (5.34) through (5.35), equation (6.14) and the induction hypothesis imply that

$$\varphi^{\mu}_{(n-1)} + \sum_{i,t} \overset{i}{A}{}^{\mu:(s)_t}_{(n-1)} \overset{i}{\Psi}_{,(s)_t} = \sum_{i} \left(\overset{i}{k}{}^{\mu}_{(n-1)} \overset{i}{\Psi} + \overset{i}{k}{}^{\mu s}_{(n-1)} \overset{i}{\Psi}_{,s} \right) + \overset{*}{O}_{n-1}\text{-function},$$

from which we derive that

$$\overline{h}^{\mu s}_{(n)\,,s} - a^{\mu}_{(n)} = \overset{*}{\varphi}{}^{\mu}_{n} + \sum_{i} \left(\overset{i}{k}{}^{\mu}_{(n-1)} \overset{i}{\Psi} + \overset{i}{k}{}^{\mu s}_{(n-1)} \overset{i}{\Psi}_{,s} \right) + \sum_{i,t} \overset{i}{k}{}^{\mu:(s)_t}_{n} \overset{i}{\Psi}_{,(s)_t} + \overset{*}{O}_n\text{-function}, \tag{6.15}$$

where $\overset{*}{q}{}^{\mu}_{n}$ is a non-singular harmonic function of the n'th order, and the $\overset{i}{k}{}^{\mu:(s)_t}_{n}$ are also of the n'th order. If we define a harmonic function, $\chi^{\mu\nu}_{n}$, by the equations following (3.15), with the φ^{μ}_{n} of that section replaced by the $\overset{*}{q}{}^{\mu}_{n}$ of this, then the $h^{\mu\nu}_{n}$ defined by

$$\begin{aligned}
h^{00}_{n} &= \overline{h}^{00}_{n}, \\
h^{0m}_{n} &= \overline{h}^{0m}_{n} - \sum_{i,t} \overset{i}{k}{}^{0:m(s)_t}_{n} \overset{i}{\Psi}_{,(s)_t} + \chi^{0m}_{n}, \\
h^{uv}_{n} &= \overline{h}^{uv}_{n} + \chi^{uv}_{n} - \tfrac{1}{2} \sum_{i} \left(\overset{i}{k}{}^{u\,v}_{n} + \overset{i}{k}{}^{v\,u}_{n} \right) \overset{i}{\Psi} - \sum_{i,t} \left(\overset{i}{k}{}^{u\,vr(s)_t}_{n} + \overset{i}{k}{}^{v:ur,s)_t}_{n} - \overset{i}{k}{}^{r:uv(s)_t}_{n} \right) \overset{i}{\Psi}_{,r(s)_t}
\end{aligned}$$

will satisfy the reduced field equations, and also

$$\overline{h}^{\mu s}_{(n)\,,s} - a^{\mu}_{(n)} = \sum_{i} \left(\overset{i}{k}{}^{\mu}_{(n)} \overset{i}{\Psi} + \overset{i}{k}{}^{\mu s}_{(n)} \overset{i}{\Psi} \right)_{,s} + \overset{*}{O}_n\text{-function}. \tag{6.16}$$

The reiteration $\overset{i}{k}{}^{A}_{(n)}$ are defined by

$$\overset{i}{k}{}^{A}_{(n)} = \overset{i}{k}{}^{;}_{(n-1)} + \overset{i}{k}{}^{\,1}_{n}.$$

THEOREM V. *It is possible to solve the reduced field equations* (6.4) *in such a way that* (6.16) *is satisfied for all n, that is so that* $\underset{(n)}{z^\mu}$ *is an* O_n*-function. The* $\underset{s}{\overset{i}{k}}{}^\mu$ *and* $\underset{s}{\overset{i}{k}}{}^{mn}_{\smile}$ *may be calculated in each approximation. If we restrict the motion of the particles by the equations,*

$$\underset{(n)}{\overset{i}{k}}{}^\mu = 0\,, \qquad \underset{(n)}{\overset{i}{k}}{}^{mn}_{\smile} = 0\,, \qquad \text{to the } n\text{'th order}\,, \tag{6.17}$$

then $\underset{(n)}{h^{\mu\nu}}$ *will satisfy the Einstein equations* to the n'th approximation.

The second part of this theorem follows from the observation that (6.17) implies that all O_n-functions are zero to the n'th approximation. Consequently, from (6.9), $\underset{(n)}{\mathfrak{G}^{\mu\nu}}$, and so $\mathfrak{G}^{\mu\nu}(\underset{(n)}{h}{}^{\alpha\beta})$, will be zero to the required order.

This theorem shows us how we may solve the field equations to any order. Let us suppose that we have solved all the lower order field equations and calculated the lower order k's. Also, let us suppose that the lower order fields satisfy (6.16) with n replaced by $n-1$. We then calculate a particular solution of the reduced field equations, $\underset{(n)}{\bar{h}^{\mu\nu}}$. If we calculate the divergence of the function, subtract $\underset{n}{a^\mu}$ from it, and add the resultant function onto $\underset{(n-1)}{z^\mu}$, we obtain $\underset{(n)}{\bar{z}^\mu}$. By using the equations of motion to the $(n-1)$'th approximations $\underset{(n-1)}{\overset{i}{k}}{}^A = 0$, we may write $\underset{(n)}{z^\mu}$ as in (6.15). We then add a harmonic function to $\underset{n}{\bar{h}^{\mu\nu}}$ to remove the higher order poles and the non-singular harmonic function from $\underset{(n)}{\bar{z}^\mu}$. This gives an $\underset{n}{h^{\mu\nu}}$ which satisfies (6.16) and defines the n'th order equations of motion.

*The proofs of Theorem II and II-*A. Both of these theorems follow as corollaries of Theorem V. Let us suppose that $\underset{(n)}{h^{\mu\nu}}$ satisfies the conditions of Theorem V and that we have introduced a mass, dipole moment and angular momentum vector for each particle into the lowest approximation. Then

$$\sum_r \underset{r}{\overset{i}{k}}{}^A = \sum_r \underset{r}{\overset{i}{k}}{}^A(\overset{i}{m}, \overset{i}{D}{}^n, \overset{i}{A}{}_n, \ldots; \overset{i}{\xi}{}^s)\,, \tag{6.18}$$

and all O_n-functions will be zero to the n'th approximation whenever the k's are zero to the same order. Let us now expand the particle parameters as λ-series,

$$\overset{i}{m} = \sum_s \lambda^s \underset{s}{\overset{i}{m}}{}^s\,, \quad \text{etc.}\,. \tag{6.19}$$

If we insert these series into $\underset{(n)}{h^{\mu\nu}}$ and collect together all terms of the r'th order, we obtain new approximation fields which satisfy the reduced field equations, (6.4), in all orders up to the n'th. This gives new approximation fields, $\underset{s}{\overset{i}{h}}{}^{\mu\nu}$, and, by expanding the $\underset{(n)}{\overset{i}{k}}{}^A$, new $\underset{s}{\overset{i}{k}}{}^A$.

Since the lowest order k's are identical to the lowest order EIH surface

integrals, we have

$$\overset{i}{k}{}^{0}_{(n)} = 4\overset{i}{m}_{,0} + \overset{i}{k}{}^{\prime 0}_{(n)}, \quad \text{etc.},$$

(6.20)

as in (4.12), (4.13) and (4.15). When we expand out the particle parameters, (6.19), we obtain

$$\overset{i}{k}{}^{0}_{s} = 4\,\overset{i}{m}_{s-1,0} + \overset{i}{k}{}^{0}_{s}(\overset{j}{m}_{t}, \ldots).$$

If we now equate the $\overset{i}{k}{}^{A}_{s}$ to zero, for all $s \leqslant n$, then all O_n-functions will be *exactly* zero. Consequently, we shall have

$$\overset{i}{k}{}^{A}_{s} = 0 \quad \text{implies} \quad \text{that} \quad \overset{*}{h}{}^{\mu s}_{r},_{s} - \overset{*}{a}{}^{\mu}_{r} = 0,$$

(6.21)

for all $r \leqslant n$. Also, from (6.9), we shall have

$$\overset{*}{\mathfrak{G}}{}^{\mu\nu}_{(n)} = \overset{*}{\mathfrak{G}}{}^{\prime\mu\nu}_{(n)} = 0.$$

Consequently, the EIH approximation equations will be satisfied. Trivially, $\overset{*}{h}{}^{\mu\nu}_{s}$ will satisfy all the conditions of Theorem II-A.

To prove Theorem II, we shall write the quasi-static field equations as

$$\begin{cases} -\frac{1}{2}\overset{*}{h}{}^{00}_{r},_{ss} + \frac{1}{2}\overset{*}{h}{}^{st}_{r},_{st} + \overset{*}{\Delta}{}^{00}_{r} = 0, \\ -\frac{1}{2}\overset{*}{h}{}^{0n}_{r},_{ss} + \overset{*}{\Delta}{}^{0n}_{r} = -\overset{*}{a}{}^{0}_{r},_{n}, \\ -\frac{1}{2}\overset{*}{h}{}^{mn}_{r},_{ss} + \overset{*}{\Delta}{}^{mn}_{r} = -\overset{*}{a}{}^{m}_{r},_{n} - \overset{*}{a}{}^{n}_{r}{}_{m} + \delta^{mn}\overset{*}{a}{}^{s}_{r},_{s}. \end{cases}$$

(6.22)

$$\overset{*}{z}{}^{\mu}_{(r)} \equiv \overset{*}{\mathfrak{g}}{}^{\mu s}_{(r)},_{s} - \overset{*}{a}{}^{\mu}_{(r)} = 0.$$

(6.23)

These are equivalent to (3.29) and (3.33), with $\mathfrak{D}^{\mu}$ replaced by z^{μ}. In precisely the same way as we proved Theorem V, we prove:

Lemma. *The field equations*, (6.22) *may be solved in such a way that*

$$\overset{*}{\mathfrak{D}}{}^{\mu}_{(n)} = \overset{*}{z}{}^{\mu}_{(n)} = \sum_{i} (\overset{i}{k}{}^{\mu}_{(n)}\overset{i}{\Psi} + \overset{i}{k}{}^{\mu s}_{(n)}\overset{i}{\Psi}_{,s}) + \overset{*}{O}_{n}\text{-function}.$$

This is completely equivalent to Theorem II, with $\overset{*}{C}{}^{\mu}_{r}$ and $\overset{*}{B}{}^{mn}_{r}$ of Section 3 replaced by the $\overset{*}{k}{}^{\mu}_{r}$ and $\overset{*}{k}{}^{mn}_{r}$ of the Lemma. The verification is trivial.

The use of the surface integrals. – One disadvantage of the method of this section is that it would seem that we have to calculate the field to the n'th order before we can calculate the equations of motion to the same approximation. However, this is not so. Corresponding to the surface integral con-

ditions of Section 3, we have

$$(6.24,a) \qquad \overset{i}{\underset{(n)}{\mathfrak{C}}}{}^{\mu} = \frac{1}{2\pi}\int^{i}\left(\underset{(n)}{\Xi}{}^{\mu s} + \frac{1}{2}\,\underset{(n)}{a}{}^{\mu}{}_{,s}\right) \mathrm{d}S_s\,,$$

$$(6.24,b) \qquad \overset{i}{\underset{(n)}{\mathfrak{A}}}{}_{m} = \varepsilon_{mrs}\frac{1}{4\pi}\int^{i}\left[\overset{i}{x}{}^{r}\left(\underset{(n)}{\Xi}{}^{su} + \frac{1}{2}\,\underset{(n)}{a}{}^{s}{}_{,n}\right) - \frac{1}{2}\,\delta^{ru}\,\underset{(n)}{a}{}^{s}\right] \mathrm{d}S_n\,.$$

From the Einstein surface integral lemma, these are zero whenever (6.2) and (6.4) are satisfied exactly. It is easily proved, using the methods of this section, that if the field equations, (6.4), are satisfied to the $(n-1)$'th order, and if the equations of motion, (6.17) are satisfied to the same order, then the surface integrals defined in (6.24) will be independent of the surfaces of integration, modulo the equations of motion. Also, as in Lemma 4, the surface integrals give the equations of motion to the n'th approximation. The proof of this is equivalent to that for Lemma 4.

The surface integrals enable us to simplify the calculations of the equations of motion to the n'th approximation. All that is necessary is to calculate the field to the $(n-1)$'th approximation by the methods of this section. This automatically gives the equations of motion to the $(n-1)$'th approximation, through Theorem V. The surface integrals of (6.24) may then be calculated using the equations of motion to the $(n-1)$'th approximation. This given the equations of motion to the n'th approximation.

7. – The δ-function techniques.

There have been many other methods used to derive the equations of motion of particles in general relativity, notably that of INFELD [9] in 1957 in which he used a modified δ-function as a source tensor, and also the Lagrangian method of Plebanski, which also uses a δ-function.

At present there is very little theoretical justification for the δ-function technique. Some of the questions which remain to be answered are as follows:

1) It is assumed that the source function is the limit of a spherically symmetric function. However, in all order past the third there is no spherically symmetric solution of the differential equations which the source function has to satisfy, namely

$$\mathfrak{T}^{\mu\nu}{}_{,\nu} = -\,\Gamma^{\mu}_{\alpha\beta}\,\mathfrak{T}^{\alpha\beta}\,,$$

[9] L. INFELD: *Rev. Mod. Phys.*, **29**, 398 (1957).

when there is more than one particle in the field. This means that the assumption that the source function is a spherically symmetric δ-function is inconsistent with the field equations.

2) It is necessary to prove a theorem equivalent to Theorems II, II-A, IV or V. This is not a trivial extension of Theorem V.

3) In the higher orders, no solution of the field equations is spherically symmetric at the particles. Now, the equations of motions as calculated by INFELD in 1957 by the δ-function technique are independent of $\overset{4}{h}{}^{mn}$. However, if the equations of motion are calculated by the EIH method they will be found to depend on the particular solution, $\overset{4}{h}{}^{mn}$, chosen. Different solutions give different equations of motion. Consequently, the δ-function technique can only give the correct equations of motion for *one* choice of $\overset{4}{h}{}^{mn}$, that is it must give a unique prescription for solving the field equations in each order. Even if it does this, it is necessary to prove that the equations of motion are correct for this field.

In spite of our seeming pessimism about the δ-function techniques, we feel that it can be put on a sound theoretical foundation. Its advantage is that the calculations are much simpler in this method than in those discussed in the earlier sections.

APPENDIX

The Einstein tensor density.

FOCK [4] has shown that if we introduce the following functions,

$$\Pi^{\mu,\alpha\beta} = \frac{1}{2g}\left(\mathfrak{g}^{\alpha\varrho}\,\mathfrak{g}^{\mu\alpha}{}_{,\varrho} + \mathfrak{g}^{\beta\varrho}\mathfrak{g}^{\mu\alpha}{}_{,\varrho} - \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta}{}_{,\varrho}\right),$$

$$\Pi_{\alpha\beta,\mu} = g_{\alpha\varrho}g_{\beta\nu}\Pi^{\mu,\varrho\nu},$$

$$y_\mu = (\log(-g)^{-\frac{1}{2}})_{,\mu}, \qquad y^\mu = g^{\mu\alpha}y_\alpha,$$

then the Einstein tensor density may be reduced to the following form

$$\text{(A.1)}\qquad \mathfrak{G}^{\mu\nu} = -\tfrac{1}{2}(-g)^{-\frac{1}{2}}\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu}{}_{,\alpha\beta} + (-g)^{\frac{1}{2}}\Pi^{\mu,\alpha\beta}\Pi_{\alpha\beta}{}^{\nu} - \tfrac{1}{4}g^{\mu\nu}\Pi_{\alpha\beta}{}^{\varrho}\mathfrak{g}^{\alpha\beta}{}_{,\varrho} -$$
$$-\tfrac{1}{2}(-g)^{\frac{1}{2}}y^\mu y^\nu + \tfrac{1}{4}\mathfrak{g}^{\mu\nu}y^\alpha y_\alpha + \tfrac{1}{2}(g^{\mu\alpha}\mathfrak{g}^{\nu\varrho}{}_{,\varrho\alpha} + g^{\nu\alpha}\mathfrak{g}^{\mu\varrho}{}_{,\varrho\alpha} -$$
$$-g^{\mu\nu}\mathfrak{g}^{\alpha\beta}{}_{,\alpha\beta}) - \tfrac{1}{2}(g^{\mu\nu}{}_{,\alpha} + g^{\mu\nu}\log(-g)_{,\alpha})\,\mathfrak{g}^{\alpha\varrho}{}_{,\varrho}\,.$$

For our purposes, it is sufficient that it should be possible to write this as

$$\mathfrak{G}^{\mu\nu} \equiv + \Phi^{\mu\nu}{}_{,s} + \Lambda^{\mu\nu} = -\tfrac{1}{2}h^{\mu\nu}{}_{,ss} + \overset{\prime}{\Lambda}{}^{\mu\nu} + \tfrac{1}{2}(g^{\nu x}\mathfrak{g}^{\mu s}{}_{,sx} + g^{\mu\alpha}\mathfrak{g}^{\nu s}{}_{,sx} - g^{\mu\nu}\mathfrak{g}^{\alpha s}{}_{,sx}) - \\ - \tfrac{1}{2}(g^{\mu\nu}{}_{,\alpha} + g^{\mu\nu}\log(-g)_{,\alpha})\mathfrak{g}^{\alpha s}{}_{,s},$$

where $\Lambda^{\mu\nu}$ and $\overset{\prime}{\Lambda}{}^{\mu\nu}$ have the properties assumed in Sections 3 and 6, respectively.

RIASSUNTO (*)

Si discutono i tre metodi principali usati per la risoluzione delle equazioni di campo quasi statiche e si mostra che alcuni teoremi devono essere dimostrati prima di poter dire che le equazioni fisiche del moto procedono dalla simmetria del campo attorno alle sorgenti. Abbiamo dimostrato questi risultati e durante lo svolgimento abbiamo mostrato che ci sono sette e soltanto sette equazioni fisiche del moto per ogni particella. Queste corrispondono alle equazioni classiche dell'energia, del moto e della quantità di moto angolare. Abbiamo anche mostrato che le equazioni di campo quasi statiche possono essere integrate senza sviluppare in serie alcuni dei parametri della particella, e senza introdurre un tensore sforzo-energia.

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On the Theory of some Čerenkovian Effects (*).

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Summary. — The field generated by a charged article in uniform straight motion is expanded into a set of evanescent waves. The expansion is valid in any half-space with no points in common with the path of the particle. The evanescent waves may impinge on the surface of an optical diffraction grating and be diffracted. Some of the diffracted waves turn out to be ordinary plane waves, which carry energy away from the grating. It is possible in this way to explain the Smith and Purcell effect and to derive some quantitative conclusions.

1. – Introduction.

A number of situations are known in which radiation can be emitted by a charged particle in uniform and straight motion [1]. It seems convenient to call all these effects « Čerenkovian effects », since they are generalizations of the well known effect discovered by Čerenkov.

A simple example of a Čerenkovian effect is the emission of radiation by a particle passing along the surface of a dielectric material at a close distance [2,3]. Another example is the radiation effect described by Smith and

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(1) For a comprehensive review see: J. V. Jelley: *Čerenkov Radiation* (London, 1958).

(2) J. G. Linhart: *Journ. Appl. Phys.*, **26**, 527 (1955).

(3) M. Danos: *Phys. Rev.*, **94**, 758 (1954); *Journ. Appl. Phys.*, **26**, 2 (1953).

PURCELL ([4]), which occurs when a charged particle passes close to the surface of a metal diffraction grating. More generally, a particle passing close to a transmission line can excite an eigenmode of the line, as was investigated by PIERCE ([5]).

In all these cases, it would seem convenient to transform the electromagnetic field generated by the particle into a set of plane waves. For, it is well known how a plane wave is refracted and reflected by the surface of a dielectric or is diffracted by a grating. A transformation of this kind has been largely applied for the calculation of ionization losses and Bremsstrahlung ([6-9]). However, such a transformation in the case of a particle travelling in empty space is only approximate. The approximation becomes better and better as $\beta = v/c$ approaches unity.

But, it is not necessary to require that the waves of the set should be ordinary plane waves. We will show that the field generated by the moving particle in empty space can be exactly transformed into a set of evanescent waves ([10]). The transformation is valid in a whole half-space having no points in common with the line of motion of the particle. The evanescent waves are all attenuated in the direction perpendicular to the boundary of the half-space.

Now evanescent waves are reflected and refracted at the plane surface of a dielectric or diffracted by a plane grating ([11]) in much the same way as ordinary waves. It, therefore, becomes an easy matter to discuss a number of Čerenkovian effects of the above mentioned type. By this approach one also obtains a very clear insight into the physical process. As an example, we will derive the formula for conventional Čerenkov radiation, starting from the case when the path of the particle is sandwiched between two dielectrics ([12]). The method of the evanescent waves will then be applied to the discussion of the Smith and Purcell effect. Of course, in this case the results will depend on the microscopic structure of the grating and will be different for different gratings. However, under some reasonable conditions, it is possible to derive some general properties of the radiation. It also becomes obvious that the effect will take place even if the metal grating is replaced by a transmission grating.

([4]) S. J. SMITH and E. M. PURCELL: *Phys. Rev.*, **92**, 1069 (1953).
([5]) J. R. PIERCE: *Journ. Appl. Phys.*, **26**, 627 (1955).
([6]) E. FERMI: *Zeits. f. Phys.*, **29**, 315 (1924).
([7]) E. J. WILLIAMS: *Proc. Roy. Soc.*, **139**, 163 (1933).
([8]) C. F. V. WEIZSÄCKER: *Zeits. f. Phys.*, **88**, 612 (1934).
([9]) W. HEITLER: *Quantum Theory of Radiation* (Oxford, 1954) p. 414.
([10]) For the properties of evanescent waves see, for instance: G. TORALDO DI FRANCIA: *Electromagnetic Waves* (New York, 1956), p. 171.
([11]) G. TORALDO DI FRANCIA: *La Diffrazione della Luce* (Torino, 1958), p. 104.
([12]) See Ref. ([3]).

2. – Expansion of the field into a set of evanescent waves.

We will use e.g.s. gaussian units. Referring to a rectangular system of co-ordinates x, y, z (unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$), we will consider a point charge e traveling in empty space with constant velocity v along the x axis. The electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ in the laboratory system will be represented by [13]

$$(1)\qquad \begin{cases} \boldsymbol{E} = \dfrac{e}{\sqrt{1-\beta^2}} \dfrac{(x-vt)\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\left[\dfrac{(x-vt)^2}{1-\beta^2} + y^2 + z^2\right]^{\frac{3}{2}}}, \\ \boldsymbol{H} = \dfrac{e\beta}{\sqrt{1-\beta^2}} \dfrac{-z\boldsymbol{j} + y\boldsymbol{k}}{\left[\dfrac{(x-vt)^2}{1-\beta^2} + y^2 + z^2\right]^{\frac{3}{2}}}, \end{cases}$$

where $\beta = v/c$. Obviously, the x dependence of the field is only through the expression $x - vt$. It is therefore expedient for the observer at rest to refer to a moving origin having the same velocity as the charge and consequently to replace $x - vt$ by x.

Further, if we assume the charge to travel at a distance $z = -a$ above the plane $z = 0$ (Fig. 1), the field at $z = 0$ will take the form

$$(2)\qquad \begin{cases} \boldsymbol{E} = \dfrac{e\beta}{\sqrt{1-\beta^2}} \dfrac{x\boldsymbol{i} + y\boldsymbol{j} + a\boldsymbol{k}}{\left[\dfrac{x^2}{1-\beta^2} + y^2 + a^2\right]^{\frac{3}{2}}}, \\ \boldsymbol{H} = \dfrac{e}{\sqrt{1-\beta^2}} \dfrac{-a\boldsymbol{j} + y\boldsymbol{k}}{\left[\dfrac{x^2}{1-\beta^2} + y^2 + a^2\right]^{\frac{3}{2}}}. \end{cases}$$

We want to represent this field by a superposition of evanescent waves.

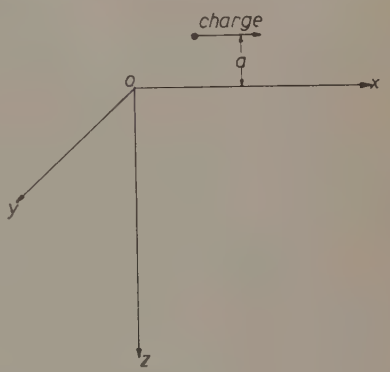

Fig. 1.

3. – As is customary in microwave theory, an electromagnetic wave will be termed a TM wave if $H_z = 0$ and a TE wave if $E_z = 0$. As is well known, TM and TE waves have different laws of reflection and refraction, hence it is convenient for our future work to consider them separately. A TM solution

(13) See *e.g.* A. SOMMERFELD: *Electrodynamics* (New York, 1952), p. 240. In this book, the sign of the magnetic field is obviously wrong.

of Maxwell's equations can be written in the form

$$(3)\qquad \begin{cases} \boldsymbol{E} = A_1\,[\,p\boldsymbol{i} + q\boldsymbol{j} - (p^2+q^2)\boldsymbol{k}]\,\exp\,[\,ik(ct - px - qy - rz)]\,, \\ \boldsymbol{H} = A_1[-q\boldsymbol{i} + p\boldsymbol{j}]\,\exp\,[\,ik(ct - px - qy - rz)]\,, \end{cases}$$

where k is real, A_1, p, q, r are complex constants and

$$(4)\qquad p^2 + q^2 + r^2 = 1\,.$$

A corresponding TE solution is obtained by replacing $\boldsymbol{E}$ by $\boldsymbol{H}$ and $\boldsymbol{H}$ by $-\boldsymbol{E}$, as is allowed by Maxwell's equations. There results

$$(5)\qquad \begin{cases} \boldsymbol{E} = A_2[q\boldsymbol{i} - p\boldsymbol{j}]\,\exp\,[\,ik(ct - px - qy - rz)]\,, \\ \boldsymbol{H} = A_2\,[\,p\boldsymbol{i} + q\boldsymbol{j} - (p^2+q^2)\boldsymbol{k}]\,\exp\,[\,ik(ct - px - qy - rz)]\,. \end{cases}$$

If the three numbers p, q, r are all real, then (3) and (5) represent ordinary plane waves having the propagation constant $|k| = 2\pi/\lambda$, and p, q, r represent the direction cosines of the direction of propagation. But if one at least of the numbers p, q, r is complex, we obtain evanescent waves.

We want the exponentials to contain x and t in the combination $x - vt$ only. Accordingly, we shall put $p = 1/\beta$. Next we shall refer to the moving origin by replacing $x - vt$ by x.

Since $1/\beta > 1$, the two numbers q, r cannot both be real, consequently we have evanescent waves. It is convenient to put

$$(6)\qquad p = \frac{1}{\beta}\,,\qquad q \text{ real}\,,\qquad r = \pm i\sqrt{\varepsilon^2 + q^2}\quad \left(\varepsilon = \frac{\sqrt{1-\beta^2}}{\beta}\right).$$

In this way, (4) is satisfied and, for $k > 0$, the waves are attenuated in the positive or negative z direction according as the lower or upper sign is chosen in the third equation. The opposite happens for $k < 0$.

By superimposing at the xy plane a continuous set of both TM and TE waves of different k and q, we can write

$$(7)\qquad \begin{cases} \boldsymbol{E} = \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}k \int_{-\infty}^{+\infty} \mathrm{d}q\,[(A_1 p + A_2 q)\boldsymbol{i} + (A_1 q - A_2 p)\boldsymbol{j} - A_1(p^2+q^2)\boldsymbol{k}]\cdot \\ \qquad\qquad \cdot \exp\left[-ik\left(\frac{x}{\beta} + qy\right)\right], \\ \boldsymbol{H} = \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}k \int_{-\infty}^{+\infty} \mathrm{d}q\,[(-A_1 q + A_2 p r)\boldsymbol{i} + (A_1 p + A_2 p r)\boldsymbol{j} - A_2(p^2+q^2)\boldsymbol{k}]\cdot \\ \qquad\qquad \cdot \exp\left[-ik\left(\frac{x}{\beta} + qy\right)\right], \end{cases}$$

where A_1, A_2 are functions of k, q.

4. – If we change the variables of integration into

$$\xi = \frac{k}{\beta}, \qquad \eta = kq \tag{8}$$

we get from (7) the Fourier integrals

$$\begin{cases} \boldsymbol{E} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \boldsymbol{B} \exp\left[-i(\xi x + \eta y)\right] \mathrm{d}\xi\, \mathrm{d}\eta\,, \\ \boldsymbol{H} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \boldsymbol{C} \exp\left[-i(\xi x + \eta y)\right] \mathrm{d}\xi\, \mathrm{d}\eta\,, \end{cases} \tag{9}$$

with

$$\begin{cases} \boldsymbol{B} = \dfrac{\beta}{|k|}\left[(A_1 pr + A_2 q)\boldsymbol{i} + (A_1 qr - A_2 p)\boldsymbol{j} - A_1(p^2+q^2)\boldsymbol{k}\right], \\ \boldsymbol{C} = \dfrac{\beta}{|k|}\left[(-A_1 q + A_2 pr)\boldsymbol{i} + (A_1 p + A_2 qr)\boldsymbol{j} - A_2(p^2+q^2)\boldsymbol{k}\right]. \end{cases} \tag{10}$$

By substituting (2) into (9) and inverting the Fourier integrals, one obtains

$$\begin{cases} \boldsymbol{B}(\xi,\eta) = \dfrac{e\gamma}{4\pi^2}\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{x\boldsymbol{i} + y\boldsymbol{j} + a\boldsymbol{k}}{[\gamma^2 x^2 + y^2 + a^2]^{\frac{3}{2}}} \exp\left[i(\xi x + \eta y)\right] \mathrm{d}x\, \mathrm{d}y\,, \\ \boldsymbol{C}(\xi,\eta) = \dfrac{e}{4\pi^2\varepsilon}\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{-a\boldsymbol{j} + y\boldsymbol{k}}{[\gamma^2 x^2 + y^2 + a^2]^{\frac{3}{2}}} \exp\left[i(\xi x + \eta y)\right] \mathrm{d}x\, \mathrm{d}y\,, \end{cases} \tag{11}$$

where $\gamma = 1/\sqrt{1-\beta^2}$.

The integrals involved in (11) belong to two fundamental types only, and it is readily derived that the following identities must hold

$$B_x(\xi,\eta) = \frac{1}{\gamma} B_y\left(\gamma\eta, \frac{\xi}{\gamma}\right), \tag{12}$$

$$C_x = 0\,, \qquad C_y = -\beta B_z\,, \qquad C_z = \beta B_y\,. \tag{13}$$

By taking into account (6), one can verify that the three equations (13) are

satisfied by the expression (10), provided that the condition holds

$$A_2 = \beta \frac{q}{r} A_1 . \tag{14}$$

Upon substitution of this condition into the first equation (10), we get

$$\boldsymbol{B} = A_1 \frac{\beta}{|k|} \left[\left(\frac{r}{\beta} + \beta \frac{q^2}{r} \right) \boldsymbol{i} + q \left(r - \frac{1}{r} \right) \boldsymbol{j} - \left(\frac{1}{\beta^2} + q^2 \right) \boldsymbol{k} \right] . \tag{15}$$

From this expression, by taking into account (6) and (8), we derive

$$\frac{B_x(\xi, \eta)}{B_y(\xi, \eta)} = \frac{1}{\gamma^2 \beta q} = \frac{1}{\gamma^2} \frac{\xi}{\eta} , \tag{16}$$

$$\frac{B_z(\xi, \eta)}{B_y(\xi, \eta)} = \frac{r}{q} = \frac{\pm i\xi \sqrt{\xi^2 + \gamma^2 \eta^2}}{|\xi| \gamma \eta} = \pm \frac{k}{|k|} \frac{i\sqrt{\xi^2 + \gamma^2 \eta^2}}{\gamma \eta} , \tag{17}$$

where the upper and lower signs correspond to the upper and lower signs of (6) respectively.

Replacing $B_y(\xi, \eta)$ in (16) by its expression (12) we can also write

$$\frac{B_y(\gamma\eta, \xi/\gamma)}{B_y(\xi, \eta)} = \frac{\xi}{\gamma \eta} . \tag{18}$$

Our task will now be to evaluate B_y and B_z from (11) and to verify whether or not they satisfy (17) and (18). If B_y and B_z satisfy (17) and (18), we can solve (15) for A_1 and (14) will give us A_2.

5. – The integrals (11) can be evaluated by means of the substitution

$$\begin{cases} x = \dfrac{1}{\gamma} \varrho \cos \theta , & y = \varrho \sin \theta , \\ \xi = \gamma \chi \cos \varphi , & \eta = \chi \sin \varphi . \end{cases} \tag{19}$$

By applying the formula from the theory of Bessel functions [14]

$$\int_0^\infty \frac{J_\nu(\chi\varrho) \varrho^{\nu+1}}{(\varrho^2 + a^2)^{\mu+1}} \, d\varrho = \frac{\chi^\mu a^{\nu-\mu}}{2^\mu \Gamma(\mu + 1)} K_{\nu-\mu}(\chi a) , \tag{20}$$

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[14] G. N. Watson: *Theory of Bessel Functions* (Cambridge, 1922), p. 418.

which holds for $-1 < R(\nu) < 2R(\mu) + \frac{3}{2}$, we find

$$(21)\qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\exp\left[i(\xi x+\eta y)\right]}{(\gamma^2 x^2+y^2+a^2)^{\frac{3}{2}}}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{\gamma}\int_0^{\infty}\frac{\varrho\,\mathrm{d}\varrho}{(\varrho^2+a^2)^{\frac{3}{2}}}\int_0^{2\pi}\exp\left[i\chi\varrho\cos(\theta-\varphi)\right]\mathrm{d}\theta =$$

$$= \frac{2\pi}{\gamma}\int_0^{\infty}\frac{J_0(\chi\varrho)\varrho\,\mathrm{d}\varrho}{(\varrho^2+a^2)^{\frac{1}{2}}} = \frac{2}{\gamma}\sqrt{\frac{2\pi\chi}{a}}\,K_{-\frac{1}{2}}(\chi a) = \frac{2}{\gamma}\sqrt{\frac{2\pi}{\gamma a}}\,(\xi^2+\gamma^2\eta^2)^{\frac{1}{4}}K_{-\frac{1}{2}}\left(\frac{a}{\gamma}\sqrt{\xi^2+\gamma^2\eta^2}\right).$$

In the same way we get

$$(22)\qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{y\exp\left[i(\xi x+\eta y)\right]}{(\gamma^2 x^2+y^2+a^2)^{\frac{3}{2}}}\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \frac{1}{\gamma}\int_0^{\infty}\frac{\varrho^2\,\mathrm{d}\varrho}{(\varrho^2+a^2)^{\frac{3}{2}}}\int_0^{2\pi}\sin\theta\exp\left[i\chi\varrho\cos(\theta-\varphi)\right]\mathrm{d}\theta = \frac{2\pi i}{\gamma}\sin\varphi\int_0^{\infty}\frac{J_1(\chi\varrho)\varrho^2\,\mathrm{d}\varrho}{(\varrho^2+a^2)^{\frac{3}{2}}} =$$

$$= \frac{2i}{\gamma}\sqrt{2\pi\chi a}\sin\varphi\,K_{\frac{1}{2}}(\chi a) = 2i\sqrt{\frac{2\pi a}{\gamma}}\,\eta(\xi^2+\gamma^2\eta^2)^{-\frac{1}{4}}K_{\frac{1}{2}}\left(\frac{a}{\gamma}\sqrt{\xi^2+\gamma^2\eta^2}\right).$$

Now since $K_{\frac{1}{2}}(x) = K_{-\frac{1}{2}}(x) = \sqrt{\pi/2x}\exp[-x]$, there follows from (11), (21) and (22)

$$(23)\qquad \begin{cases} B_y = \dfrac{ie\gamma}{2\pi}\,\eta\,\dfrac{\exp\left[-(a/\gamma)\sqrt{\xi^2+\gamma^2\eta^2}\right]}{\sqrt{\xi^2+\gamma^2\eta^2}}, \\[2ex] B_z = \dfrac{e}{2\pi}\exp\left[-\dfrac{a}{\gamma}\sqrt{\xi^2+\gamma^2\eta^2}\right]. \end{cases}$$

The identity (18) is satisfied by these expressions.

The identity (17) is satisfied too, provided that the lower sign is chosen for $k > 0$ and the upper sign for $k < 0$. This means that the evanescent waves must be attenuated in the positive direction of the z axis, *i.e.* away from the particle.

6. – From (15) and (23), we can evaluate A_1

$$(24)\qquad A_1 = -\frac{e}{2\pi}\,\frac{|k|\beta}{1+\beta^2 q^2}\exp\left[-|k|a\sqrt{\varepsilon^2+q^2}\right],$$

and from (14)

$$A_2 = -\frac{iek}{2\pi}\,\frac{\beta^2 q}{(1+\beta^2 q^2)\sqrt{\varepsilon^2+q^2}}\exp\left[-|k|a\sqrt{\varepsilon^2+q^2}\right]. \tag{25}$$

Substituting these expressions into (3) and (5), we can write the fields of the component waves. It is convenient to combine the terms with equal and opposite k and then to interpret k as a positive number (physical wave number). We thus obtain for the field of a TM wave of the set

$$\begin{cases} \boldsymbol{E} = \dfrac{ek}{\pi}\exp\left[-k(a+z)\sqrt{\varepsilon^2+q^2}\right]\cdot \\ \qquad\qquad \cdot\left\{-\dfrac{\sqrt{\varepsilon^2+q^2}}{1+\beta^2q^2}(\boldsymbol{i}+\beta q\boldsymbol{j})\sin\left[k\left(ct-\dfrac{1}{\beta}x-qy\right)\right]+\right. \\ \qquad\qquad \left.+\boldsymbol{k}\dfrac{1}{\beta}\cos\left[k\left(ct-\dfrac{1}{\beta}x-qy\right)\right]\right\}\mathrm{d}k\,\mathrm{d}q\,, \\ \boldsymbol{H} = \dfrac{ek}{\pi}\exp\left[-k(a+z)\sqrt{\varepsilon^2+q^2}\right]\dfrac{\beta q\boldsymbol{i}-\boldsymbol{j}}{1+\beta^2q^2}\cos\left[k\left(ct-\dfrac{1}{\beta}x-qy\right)\right]\mathrm{d}k\,\mathrm{d}q\,, \end{cases} \tag{26}$$

and for the field of a TE wave

$$\begin{cases} \boldsymbol{E} = \dfrac{ek}{\pi}\exp\left[-k(a+z)\sqrt{\varepsilon^2+q^2}\right]\dfrac{\beta q}{(1+\beta^2q^2)\sqrt{\varepsilon^2+q^2}}\cdot \\ \qquad\qquad \cdot(\beta q\boldsymbol{i}-\boldsymbol{j})\sin\left[k\left(ct-\dfrac{1}{\beta}x-qy\right)\right]\mathrm{d}k\,\mathrm{d}q\,, \\ \boldsymbol{H} = -\dfrac{ek}{\pi}\exp\left[-k(a+z)\sqrt{\varepsilon^2+q^2}\right]\cdot \\ \qquad\qquad \cdot q\left\{\dfrac{\beta(\boldsymbol{i}+\beta q\boldsymbol{j})}{1+\beta^2q^2}\cos\left[k\left(ct-\dfrac{1}{\beta}x-qy\right)\right]+\right. \\ \qquad\qquad \left.+\dfrac{\boldsymbol{k}}{\sqrt{\varepsilon^2+q^2}}\sin\left[k\left(ct-\dfrac{1}{\beta}x-qy\right)\right]\right\}\mathrm{d}k\,\mathrm{d}q\,. \end{cases} \tag{27}$$

Thus, we have completed the expansion of the field generated by the moving point charge into an infinite set of evanescent waves, attenuated in the positive direction of z.

Conversely, one can verify that addition of the two fields (26), (27) and integration with respect to k and q, gives back the field (1) for $z > -a$. The integration can be performed by elementary functions. For $z \leqslant -a$ the integral diverges. This means that the representation of the field (1) in terms of the evanescent waves (26), (27) is valid only in the half-space $z > -a$.

In the half-space $z < -a$ one must employ a second set of evanescent waves, attenuated in the negative direction of z. Neither one of these representations is valid for $z = a$, *i.e.* for the plane containing the particle.

7. – Čerenkov radiation.

It is evident how the set (26), (27) can be used to discuss the radiation from a charged particle traveling with uniform velocity parallel to the surface of a dielectric. Let the two half-spaces $z < 0$ and $z > 0$ be vacuum and a homogeneous dielectric respectively. Each evanescent wave produced by the particle is incident on the surface $z = 0$ and there gives rise to a reflected wave and a refracted wave. The reflected wave is an evanescent wave, attenuated in the negative direction of z. The refracted wave has the direction cosines $p' = p/n$, $q' = q/n$, $r' = \sqrt{1 - p'^2 - q'^2}$, where n represents the refractive index of the dielectric. If $n\beta > 1$ there are waves for which these direction cosines are all real. Consequently, some of the refracted waves are ordinary plane waves which carry radiation away from the surface. We will not dwell on this case which was treated by LINHART [2] with a different technique, and will briefly show how the theory of the Čerenkov effect can easily be worked out by means of our expansions.

We will assume that the homogeneous dielectric occupies both half-spaces $z > 0$, $z < -2a$ and the particle moves in the empty layer in between. Clearly, the Čerenkov case will be obtained by making a very small, as was pointed out by DANOS [3]. The dielectric will be assumed to be dispersive so that the refractive index will be a function $n(k)$ of k.

The incident waves will be the waves (26), (27) for the surface $z = 0$ and a corresponding set of waves attenuated in the negative direction of z, for the surface $z = -2a$. To satisfy the conditions of continuity, we shall have to add two other sets of evanescen twaves (multiply-reflectedw aves), attenuated in the positive and negative directions of z respectively. A moment of reflection shows that for $a \to 0$ these two sets will have equal E_x, E_y and equal and opposite H_x, H_y. This means that for a given frequency, the tangential components of $\boldsymbol{H}$ on the plane $z = 0$ tend to become identical to those of the waves (26), (27). By the condition of continuity, these same values of the tangential components of $\boldsymbol{H}$ will hold for the refracted waves.

Let us now consider the incident wave having $q = 0$. By (26), (27), its magnetic field on the plane $z = 0$ for a very small a will be given by

$$\boldsymbol{H} = -\frac{ek}{\pi}\boldsymbol{j}\cos\left[k\left(ct - \frac{1}{\beta}x\right)\right]\mathrm{d}k\,\mathrm{d}q\,. \tag{28}$$

The refracted wave will have the form

$$\boldsymbol{H} = -\frac{ek}{\pi}\boldsymbol{j}\cos\left[k'\left(\frac{c}{n}t - p'x - r'z\right)\right]\mathrm{d}k\,\mathrm{d}q\,. \tag{29}$$

Continuity at $z=0$ requires $k'=nk$ and $p'=p/n=1/\beta n$. The refracted wave travels in a direction which makes an angle θ with the axis given by

$$\cos\theta = \frac{1}{\beta n}\,. \tag{30}$$

This is the well-known formula for Čerenkov radiation.

We are now led by symmetry to surmise that a wave similar to (29) will travel in each direction which makes an angle θ with the x axis. For, the plane cut within which the particle travels, is only ideal and can have any orientation. We, therefore, expect a set of waves like (29) to appear in the dielectric and to envelope a cone with semiaperture $\pi/2-\theta$. However, we will proceed in a more rigorous manner by maintaining the small cut and taking into account the effect of all the refracted waves at a point of the xz plane. If after this the thickness of the cut is made to vanish, the point considered will simply become a general point of the dielectric.

Let us denote by $\boldsymbol{H}(k, p, q)$ the amplitude and by φ the phase of the refracted wave generated by the incident wave k, p, q. The resultant magnetic field at the point x, 0, z of the dielectric will be given by

$$\boldsymbol{H} = \int_0^{k_m}\mathrm{d}k\int_{-\infty}^{+\infty}\boldsymbol{H}(k, p, q)\exp\left[ik(ct - px - z\sqrt{n^2-p^2-q^2}) + i\varphi\right]\mathrm{d}q\,, \tag{31}$$

where the real part is understood. Here k_m indicates the highest value of k for which $n\beta \geqslant 1$ and the refracted wave $p'=1/n\beta$, $q'=0$ is still a real plane wave. Since we are only interested in the value of the field for very large values of z, we can apply the principle of stationary phase ([15]) to (31) and obtain for the real part of $\boldsymbol{H}$

$$\boldsymbol{H} = \sqrt{\frac{2\pi}{z}}\int_0^{k_m}\boldsymbol{H}(k, p, 0)\sqrt{\frac{n\sin\theta}{k}}\cdot \cos\left[k(ct - xn\cos\theta - zn\sin\theta) + \varphi_0 + \frac{\pi}{4}\right]\mathrm{d}k\,, \tag{32}$$

([15]) See for instance, ref. ([10]), p. 36.

where $\gamma_0 = \varphi(q = 0)$. Now from (29) we have $\boldsymbol{H}(k, p, 0) = ek\boldsymbol{j}/\pi$ and $\varphi_0 = -\pi$, hence, we find

$$H_y = e\sqrt{\frac{2}{\pi z}}\int_0^{k_m}\sqrt{kn\sin\theta}\,\cos\left[k(ct - xn\cos\theta - zn\sin\theta) - \frac{3\pi}{4}\right]dk\,. \tag{33}$$

By a perfectly similar argument we can evaluate the electric field and, since $\boldsymbol{E}(k, p, 0)$ is perpendicular to both $\boldsymbol{H}(k, p, 0)$ and the direction θ, further $\boldsymbol{E}(k, p, 0) = \boldsymbol{H}(k, p, 0)/n$, we easily find

$$E_x = e\sqrt{\frac{2}{\pi z}}\int_0^{k_m}\sqrt{\frac{k}{n}}\,\sin^{\frac{3}{2}}\theta\,\cos\left[k(ct - nx\cos\theta - nz\sin\theta) - \frac{3\pi}{4}\right]dk\,. \tag{34}$$

By forming the z component of the Poynting vector $S_z = cE_xH_y/4\pi$ and integrating with respect to time from $-\infty$ to $+\infty$ we obtain the total energy escaped through a unit area perpendicular to z. Multiplying by $2\pi zl$, we have the energy W radiated by the particle while traveling a distance l. Therefore, we shall write

$$W = \frac{ce^2l}{\pi}\int_{-\infty}^{+\infty}dt\int_0^{k_m}dk\sqrt{\frac{k}{n}}\,\sin^{\frac{3}{2}}\theta\,\cos\left[k(ct - nx\cos\theta - nz\sin\theta) - \frac{3\pi}{4}\right]\cdot \tag{35}$$

$$\cdot\int_0^{k_m}d\bar{k}\sqrt{\bar{k}\bar{n}\sin\bar{\theta}}\,\cos\left[\bar{k}(ct - \bar{n}x\cos\bar{\theta} - \bar{n}z\sin\bar{\theta}) - \frac{3\pi}{4}\right],$$

where the notations are used $\bar{n} = n(\bar{k})$, $\theta = \theta(k)$. By performing first the integration with respect to t and recalling that

$$\begin{cases}\displaystyle\int_{-\infty}^{+\infty}\cos(k\xi)\cos(\bar{k}\xi)\,d\xi = \pi\left[\delta(k - \bar{k}) + \delta(k + \bar{k})\right], \\ \displaystyle\int_{-\infty}^{+\infty}\sin(k\xi)\sin(\bar{k}\xi)\,d\xi = \pi\left[\delta(k - \bar{k}) - \delta(k + \bar{k})\right],\end{cases} \tag{36}$$

we obtain

$$W = e^2l\int_0^{k_m}k\sin^2\theta\,dk = e^2l\int_0^{k_m}k\left(1 - \frac{1}{n^2\beta^2}\right)dk\,. \tag{37}$$

This formula coincides with that given by FRANK and TAMM ([16]) in their theory of the, Čerenkov effect.

8. – Smith and Purcell effect.

A perfectly similar argument can be carried out when the plane $z = 0$ represents a diffraction grating with the rulings parallel to y, as in the experiment of SMITH and PURCELL ([4]). For convenience we shall refer to the case when the grating acts by transmission. The case of the reflecting grating is obtained from this by a simple reflection of all the diffracted waves at the plane $z = 0$.

If the direction cosines of the incident wave are p, q, r the direction cosines p', q', r' of the diffracted wave of the m-th order are obtained by ([17])

$$p' = p - \frac{2\pi m}{kd}, \qquad q' = q, \tag{38}$$

where d represents the spacing of the grating. With regard to diffraction there is no fundamental difference between an ordinary plane wave and an evanescent wave. The diffraction of a real plane wave can give rise to evanescent waves ([18]) and vice-versa.

In the case of the evanescent waves (26), (27), having $p = 1/\beta$ it is possible for

$$p' = \frac{1}{\beta} - \frac{2\pi m}{kd} \tag{39}$$

to be less than 1. In this case, if $p'^2 + q^2 < 1$, the diffracted wave is a real plane wave, and carries energy away from the system. The directions of all the m-th order waves of a given frequency make an angle θ with the x axis given by $\cos\theta = p'$. From (39) there follows the expression of the wavelength

$$\lambda = \frac{d}{m}\left(\frac{1}{\beta} - \cos\theta\right) \tag{40}$$

which is an obvious generalization of that given by SMITH and PURCELL for the first order.

Let us now consider the whole set of waves (26), (27). By adding the TM and TE contributions, each wave of the set will be specified by the values

([16]) I. FRANK and I. G. TAMM: *Compt. Rend. Acad. Sci. URSS*, **14**, 109 (1937).
([17]) See ref. ([11]), p. 107.
([18]) See ref. ([11]), p. 108, 114.

of k, p, q. Let $\boldsymbol{E}(k, p, q)$ denote the electric-field amplitude and φ the phase of the m-th order diffracted wave, generated by the incident wave k, p, q. For the sake of simplicity, we shall confine ourselves to the plane of symmetry xz. The resultant field generated at the point x, 0, z by the m-th spectrum, taken between the values k_1, k_2 of k can be written as

$$\boldsymbol{E} = \int_{k_1}^{k_2} \mathrm{d}k \int_{-\infty}^{+\infty} \boldsymbol{E}(k, p, q) \exp\left[ik(ct - p'x - r'z) + i\varphi\right] \mathrm{d}q\,, \tag{41}$$

where $p' = \cos\theta$ is given by (39) and $r' = \sqrt{1 - p'^2 - q^2}$. We will assume that k_1 and k_2 are so close to one another as to practically correspond to the same value of θ. For very large values of z we can apply the principle of stationary phase and get for the real part of the field (41)

$$\boldsymbol{E} = \int_{k_1}^{k_2} \boldsymbol{E}(k, p, 0) \sqrt{\frac{2\pi \sin\theta}{kz}} \cos\left[k(ct - x\cos\theta - z\sin\theta) + \varphi_0 + \frac{\pi}{4}\right] \mathrm{d}k\,. \tag{42}$$

A similar expression is valid for the magnetic field, with an amplitude $\boldsymbol{H}(k, p, 0)$ equal to $\boldsymbol{E}(k, p, 0)$ and perpendicular to it and to the direction of propagation of the diffracted wave. By forming the Poynting vector and integrating with respect to t between $-\infty$ and $+\infty$, we get the total energy U which goes through a unit surface perpendicular to the direction of propagation θ, at a large distance z from the grating

$$U = \frac{c}{2z} \int_{-\infty}^{\infty} \mathrm{d}t \int_{k_1}^{k_2} E(k, p, 0) \sqrt{\frac{\sin\theta}{k}} \cos\left[k(ct - x\cos\theta - z\sin\theta) + \varphi_0 + \frac{\pi}{4}\right] \mathrm{d}k \cdot \tag{43}$$

$$\cdot \int_{k_1}^{k_2} E(\bar{k}, \bar{p}, 0) \sqrt{\frac{\sin\bar{\theta}}{\bar{k}}} \cos\left[\bar{k}(ct - x\cos\bar{\theta} - z\sin\bar{\theta}) + \bar{\varphi}_0 + \frac{\pi}{4}\right] \mathrm{d}\bar{k}\,.$$

We now recall (36), and perform the t integration, then put $k_2 - k_1 = \mathrm{d}k$, obtaining

$$U = \frac{\pi}{2z} \frac{\sin\theta}{k} \left[E(k, p, 0)\right]^2 \mathrm{d}k\,. \tag{44}$$

From (26) we see that the amplitude of the incident wave k, p, 0 is $(ek/\pi) \exp[-ka\varepsilon]$. Therefore, we can write for the amplitude of the m-th order diffracted wave

$$E(k, p, 0) = \delta_m \frac{e\bar{k}}{\pi} \exp[-ka\varepsilon]\,, \tag{45}$$

where δ_m denotes a number which depends on the structure of the grating and on k. Upon substitution of (45) into (44), we get

$$U = \frac{e^2 k}{2\pi z} \delta_m^2 \sin\theta \exp\left[-2ka\varepsilon\right] \mathrm{d}k \,. \tag{46}$$

The case of the grating does not present the cylindrical symmetry of the Čerenkov case. However, to get an order of magnitude of the total energy radiated by the particle while traveling the path length l, we may simply multiply (46) by $\pi z l \sin\theta$ and integrate with respect to k. We thus see that, if both the factor δ_m^2 and the exponential have the order of magnitude of unity, the total radiation is of the same order of magnitude as that of the Čerenkov effect in a medium having the dispersion law (39) for an incident wave with $p = 1/\beta$.

9. – If instead of one particle, there is a steady surface current of N particles per second, and if the spacings of the particles are sufficiently at random for the emission to be considered incoherent, the power P flowing across the unit surface perpendicular to the direction θ is found from (46) to be

$$P = N \frac{e^2}{2\pi z} \delta_m^2 k \sin\theta \exp\left[-2ka\varepsilon\right] \mathrm{d}k \,. \tag{47}$$

Since this power comes from a distance $z/\sin\theta$ along the ray, the unit surface perpendicular to the ray corresponds to a solid angle $\sin^2\theta/z^2$. Further, if D denotes the transversal width of the surface current, the apparent area of emission from the grating is $D(z/\sin\theta)\,\mathrm{d}\theta$. Hence, the brightness of the grating, or the power per unit solid angle, per unit apparent emission surface will be given by

$$B = \frac{Ne^2}{2\pi D} \delta_m^2 k \left|\frac{\mathrm{d}k}{\mathrm{d}\theta}\right| \exp\left[-2ka\varepsilon\right] . \tag{48}$$

Eliminating k by means of (39), where $p' = \cos\theta$ and denoting by $I = Ne/D$ the density of the surface current, we can also write

$$B = 2\pi I e \delta_m^2 \frac{m^2}{d^2} \frac{\sin\theta\, \beta^3}{(1-\beta\cos\theta)^3} \exp\left(-4\pi m \frac{a}{d} \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\right) . \tag{49}$$

It should be noted that expression (49) holds for the m-th spectrum, considered isolated from the other spectra. In practice there will be a certain superposition of different spectra.

To discuss our results, we have to make some assumptions about δ_m^2. This number represents the intensity of the m-th spectrum diffracted by the grating and depends both on the structure of the grating and on the parameters β

and θ. However, one can assume that by proper shaping of the profile of the rulings, δ_m^2 can be rendered very close to 1 for the direction and the wavelength considered. This is certainly true for the case of an ordinary reflection grating working with real plane waves (blazed gratings) and is probably true also when the incident wave is evanescent. We will, therefore, stipulate that the grating used is always a « good » one for the angle and wave length considered, so that δ_m is of the order of unity and does not vary much with θ.

It is apparent from (49) that in order to obtain as much brightness as possible, one should make a very small. In other words, the particles should travel very close to the grating. The exponential attenuation with increasing a is the most severe limiting factor to the brightness. To reduce its effect, one should make β as close to 1 as is feasible. In practice both the highest value of β and the lowest value of a will be set by the accelerating and collimating equipment available.

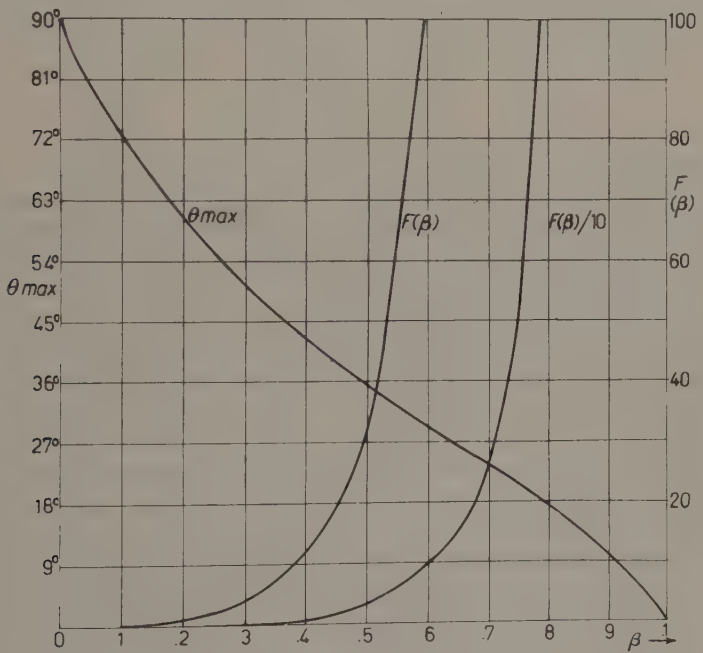

Fig. 2.

Once the values of the parameters β, a, d are fixed, there is a value $\theta_{\max}$ of the angle θ for which brightness is a maximum. Let us first investigate the ideal case when $a = 0$. It is found from (49) that in this case, $\theta_{\max} = \arccos[(\sqrt{1+24\beta^2}-1)/4\beta]$. By substituting this value into (49), one can

write for the value of B corresponding to θ_{max}

$$B_{max} = I_{\mu A/mm}\,\delta_m^2\,\frac{m^2}{(d_{mm})^2}\,F(\beta)\ \ \mathrm{erg/s\ mm^2\ sr}\,, \tag{50}$$

where the current density I is measured in μA/mm, d in μm, and $F(\beta)$ is a function of β only. The values of θ_{max} and $F(\beta)$ are plotted against β in Fig. 2. From the value of θ_{max} and d one can calculate the corresponding value of λ by means of (40). The order of magnitude of λ will be about the same as that of d.

In practice a cannot be exactly zero and our evaluation of the brightness is much too optimistic. The exponential factor in (49) will reduce by some orders of magnitude the brightness as given by (50), especially for very small values of d or λ. It will also shift θ_{max} towards higher values. A discussion of (49) shows that $\theta_{max} \lesssim 90°$ according as $4\pi m(a/d)\sqrt{1-\beta^2} \lesssim 3$. Therefore, when a is not very small compared to d, maximum radiation occurs in a backward direction. This is a feature unknown for Čerenkov radiation.

10. – Conclusion.

We have shown that the field of a point charge in uniform motion can always be considered as the superposition of an infinite set of evanescent waves. The expansion is valid in a whole half-space which does not contain the line of motion of the particle.

The representation of the field as a set of evanescent waves considerably facilitates the theoretical investigation of some Čerenkovian effects and gives an insight into their physical meaning. This procedure seems particularly useful in the case of the Smith and Purcell effect. Some previous approaches ([4,19,20]) interpreted this radiation as being produced by the electrostatic image of the point charge given by the grating, which acts as a travelling and oscillating dipole. It seems very difficult to get from this approach a rigorous theory, holding for a general type of grating. Our approach instead confronts us with a well defined mathematical problem, namely with the problem of diffraction of an evanescent wave by the grating. This problem, of course, can only be solved when the structure of the particular grating employed is known. In this connection, we point out that the evanescent wave approach can suggest the best type of grating to employ for a given purpose.

([19]) A. Kastler: *Suppl. Nuovo Cimento*, **3**, 761 (1955).

([20]) K. Ishiguro and T. Tako: *Fifth Meeting and Conference of the International Commission of Optics* (Stockholm, 1959; to be published).

For instance, if we want to estract from the particle as much energy as possible, we are interested in having a great value for δ_m. It is interesting to note that when both the incident and reflected waves are evanescent, the grating can show a sort of resonance ([21]) giving $\delta_m \gg 1$. It will be worthwhile to investigate whether a similar condition can be obtained when the incident wave is evanescent and the diffracted wave is an ordinary travelling wave.

It will also be of interest to evaluate the reaction which the reflected or diffracted waves exert upon the point charge. It should be possible in this way to assess the total power radiated.

([21]) A. L. CULLEN: *Proc. I.E.E.*, **101**, 225 (1954).

RIASSUNTO

Si mostra come il campo elettromagnetico generato da una particella carica in moto rettilineo uniforme possa essere rappresentato da un insieme di onde evanescenti. La rappresentazione è solida in tutto un semispazio non avente punti a comune con la traiettoria della particella. Le onde evanescenti possono incidere sulla superficie di un reticolo di diffrazione ed essere diffratte. Una parte delle onde diffratte è composta di ordinarie onde piane, che si propagano e trasportano energia lontano dal reticolo. È possibile in questo modo spiegare l'effetto Smith-Purcell e arrivare ad alcune utili conclusioni quantitative.

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Photoproduction of Pions on Pions.

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Summary. — Photoproduction of pions on pions is computed by means of the Cini-Fubini version of the Mandelstam technique. The problem is reduced to the resolution of a Fredholm integral equation containing an arbitrary multiplicative constant which could be determined by extrapolation of experimental cross-sections. Under the assumption of a sharp pion pion resonance we derive approximate solutions which exhibit the same resonant behaviour. These results will be soon applied to Compton scattering on pions and nucleons and photoproduction on the nucleons.

1. – Introduction.

The assumption of a double dispersion relation [1] for a two body—two body matrix element leads to interesting consequences when it is combined with the unitarity condition. It is hoped that the approximation in neglecting three body states is valid in a limited range of energy. This is being applied to pion pion [2-4], pion nucleon [5,6] and nucleon-nucleon [5,7] interaction.

(*) On leave of absence from the Laboratoire de Physique, Ecole Normale Supérieure, Paris, and Faculté des Sciences de Bordeaux.

(1) S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).

(2) G. F. Chew and S. Mandelstam: UCRL 8728, to be published.

(3) M. Cini and S. Fubini: to be published.

(4) W. Frazer and J. Fulco: *Phys. Rev. Lett.*, **2**, 365 (1959) and UCRL 8880, to be published.

(5) G. F. Chew: *Report at the Kiev International Conf. on High Energy Phys.* (1959).

(6) J. Bowcock and D. Lurié: private communication.

(7) W. Frazer and J. Fulco: UCRL 8806, to be published in *Phys. Rev.*

Examination of the processes involving photons, such as pion production on nucleons or Compton scattering on nucleons shows that a necessary intermediate step to apply the Mandelstam technique to these reactions is the calculation of the process:

$$\gamma + \pi \to \pi + \pi \ (^*) .$$

The interest of this reaction also lies in its simplicity, due to the limitation of possible states by conservation laws and symmetry properties. Assuming a known pion pion $J=1$, $T=1$ phase shift, one can reduce the determination of the matrix element to the solution of a single Mushkelishvili-Omnès integral equation which can be transformed into a Fredholm equation containing an arbitrary multiplicative constant in the inhomogeneous term. There is no way of determining this arbitrary factor in this approach because the electromagnetic coupling constant of pions never appears and since, in this process, the isoscalar part of the photon alone is acting, there is no possibility of using a low energy limit property.

In the first sections we give the rigorous treatment of this problem, following the simplified version of the Mandelstam technique proposed by CINI and FUBINI ([8]). The last section is concerned with approximate solutions of the equation and discussion of the results under the assumption of a pion pion resonance.

2. – Structure of the matrix elements. Kinematics.

We define $p_1 p_2 p_3$ as the incoming four momenta of the pions (**), $k = -p_1 - p_2 - p_3$ as the incident momentum of the photon. We define channel *one* by the reaction $1+\gamma \to 2+3$, and so on (Fig. 1).

We introduce, as usual, the quantities

$$(1) \qquad \begin{cases} s_1 = -(p_2 + p_3)^2 , \\ s_2 = -(p_1 + p_3)^2 , \\ s_3 = -(p_1 + p_2)^2 ; \end{cases}$$

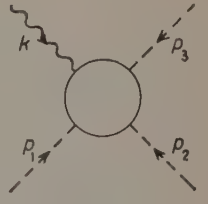

Fig. 1.

on the energy shell $s_1 + s_2 + s_3 = 3\mu^2$.

(*) This problem is also studied by H. S. WONG: private communication; and *Bull. Am. Phys. Soc.*, **4**, 407 (1959). G. BONNEVAY (private communication) has also done some work on this subject.

(**) It is useless, as will become clear later, to indicate the charge indices of the pions.

([8]) M. CINI and S. FUBINI: *Lectures on Two Dimensional Dispersion Theory* (CERN, unpublished).

From Lorentz invariance and gauge invariance the matrix element has the form (9):

$$\frac{1}{2i}\varepsilon_{\lambda\mu\nu\varrho}\frac{p_{1\mu}p_{2\nu}p_{3\varrho}e_\lambda}{4[p_{10}p_{20}p_{30}k_0]^{\frac{1}{2}}}F(s_1, s_2, s_3)\ ,$$

where e is the photon polarization.

Application of G-invariance (*) shows that only the isoscalar part of the photon contributes. Since the $T=0$ three pion state is completely antisymmetric in isospace, it follows from the boson character of the pions that the function $F(s_1s_2s_3)$ is completely symmetrical. In the reaction $\gamma+\pi\to\pi+\pi$, the two final pions are in an isospin state $T=1$; their relative orbital angular momentum is *odd*.

We now consider channel one (the crossed reactions are exactly the same due to the symmetries of F). Then the invariants can be written in the centre of mass system:

$$(3)\qquad \begin{cases} s_1=(\omega_p+p)^2=4\omega_q^2\ , \\ s_2=\mu^2-2p\omega_q+2pq\cos\theta\ , \\ s_3=\mu^2-2p\omega_q-2pq\cos\theta\ , \end{cases}$$

where p is the photon momentum, q is the relative momentum of the final pions, and θ the scattering angle.

The matrix element becomes

$$(4)\qquad A_e(q,\Omega_q)=-\frac{(e,p,q)}{4\sqrt{p\omega_p}}F(s_1, s_2, s_3)\ .$$

3. – Mandelstam representation.

From G-conjugation the intermediate states containing an odd number of pions are forbidden, as in the pion pion scattering. In particular the one pion state is missing so that there is no pole in the scattering amplitude.

We now assume for the scalar function $F(s_1s_2s_3)$ a Mandelstam representation, containing a single weight function, due to the above symmetry properties:

$$(5)\qquad F(s_1s_2s_3)=f(s_1s_2)+f(s_2s_3)+f(s_3s_1)\ ,$$

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(9) P. FEDERBUSH, M. L. GOLDBERGER and S. B. TREIMAN: *Phys. Rev.*, **112**, 642 (1958).

(*) G operation is a combination of charge conjugation and charge symmetry.

where

$$f(s_1 s_2) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \int_{(2\mu)^2}^{\infty} \frac{\varrho(\alpha, \beta)\, d\alpha\, d\beta}{(\alpha - s_1)(\beta - s_2)} .$$

The questions of convergence of the integrals are disregarded for the moment.

Actually, as was noticed quite generally by CINI and FUBINI (8), the two variables α and β cannot reach simultaneously their lower limit, corresponding to the two pion state. More precisely, if s_1 lies between $(2\mu)^2$ and $(4\mu)^2$, the only possible intermediate state in channel I is a two pion state—the three pion state being forbidden (Fig. 2).

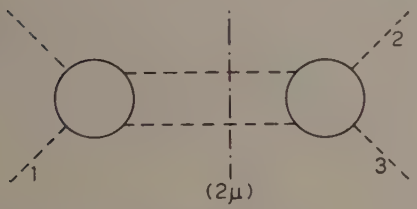

Fig. 2.

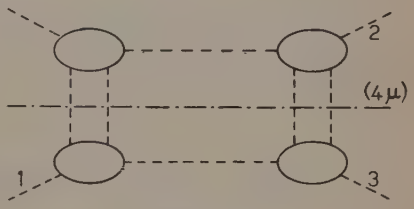

Fig. 3.

Then the minimum value of s_2 can be obtained by looking at channel II. It is easily seen that the lowest intermediate state contains at least four pions (Fig. 3).

Therefore

$$f(s_1 s_2) = \frac{1}{\pi} \int_{(2\mu)^2}^{(4\mu)^2} d\alpha \int_{(4\mu)^2}^{\infty} d\beta \frac{\varrho(\alpha, \beta)}{(\alpha - s_1)(\beta - s_2)} + \frac{1}{\pi} \int_{(4\mu)^2}^{\infty} d\alpha \int_{(2\mu)^2}^{(4\mu)^2} d\beta \frac{\varrho(\alpha, \beta)}{(\alpha - s_1)(\beta - s_2)} +$$

$$+ \frac{1}{\pi} \int_{(4\mu)^2}^{\infty} d\alpha \int_{(4\mu)^2}^{\infty} d\beta \frac{\varrho(\alpha, \beta)}{(\alpha - s_1)(\beta - s_2)} .$$

The first term of the right hand side has a strong dependence on s_1 and a weak dependence on s_2. The second term has a strong dependence on s_2 and a weak dependence on s_1. The last term has a weak dependence on both variables. Combining the three functions f one can rewrite (5) as:

$$F(s_1 s_2 s_3) = g(s_1 Z_1) + g(s_2 Z_2) + g(s_3 Z_3) ,$$

where

$$Z_1 = s_2 - s_3 ; \qquad Z_2 = s_3 - s_1 ; \qquad Z_3 = s_1 - s_2 .$$

$g(s, Z)$ has a strong dependence on s and a weak dependence on Z; it is an even function of Z because F is completely symmetrical with respect to $s_1 s_2 s_3$.

We now neglect the Z dependence; the error is of the order Z^2 (*). Then (5) becomes:

$$F(s_1 s_2 s_3) - \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\varrho(s')\,ds'}{s'-s_1} + \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\varrho(s')\,ds'}{s'-s_2} + \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\varrho(s')\,ds'}{s'-s_3} \,. \tag{6}$$

In the case of the pion pion interaction, this treatment has been shown by CINI and FUBINI (3) to be completely equivalent to the Chew-Mandelstam approach (2) in which the corresponding approximation are made later.

We provisionally assume that the integrals appearing in (6) are convergent; actually we shall see later that the final results are equivalent, with or without subtraction.

4. – Derivation of the basic integral equation.

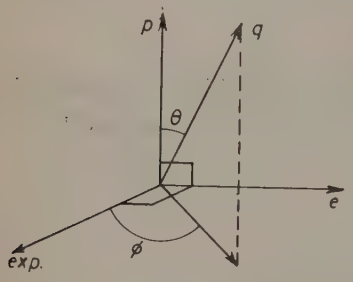

Fig. 4.

In order to apply the unitarity condition to the lowest angular momentum amplitude, we have to extract the P wave part of $A_e(q, \Omega_q)$. In the frame given in Fig. 4 angles θ and φ are defined. Then $(\boldsymbol{e}, \boldsymbol{p}, \boldsymbol{q}) = pq \sin\theta \cos\varphi$ represents a P wave with axis $\boldsymbol{e}\times\boldsymbol{p}$. The P wave part of the matrix element has the same axis.

With the help of the form (6), we can write the scattering amplitude:

$$A_e(q, \Omega_q) = \frac{pq \sin\theta \cos\varphi}{4\sqrt{p\omega_p}} \left\{ \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho(s)\,ds}{s - 4q^2 - 4\mu^2 - i\varepsilon} + \right. \tag{7}$$

$$\left. + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho(s)\,ds}{s - \mu^2 + 2p\omega_q - 2pq\cos\theta} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho(s)\,ds}{s - \mu^2 + 2p\omega_q + 2pq\cos\theta} \right\} .$$

We define the P wave amplitude as:

$$A(q^2) = \frac{3}{4\pi}\int A_e(q, \Omega_q) \sin\theta \cos\varphi \, d\Omega \tag{8}$$

(*) If it turned out that the $J = 3$ states are important this would not be permissible.

and obtain the following representation for $A(q^2)$:

$$A(q^2) = \frac{pq}{4\sqrt{p\omega_p}} B(q^2),$$

with $B(q^2)$ given by

$$(9) \qquad B(q^2) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho(s)\,\mathrm{d}s}{s - 4q^2 - 4\mu^2 - i\varepsilon} + \frac{3}{4\pi}\int_{-1}^{+1} \sin^2\theta\,\mathrm{d}\cos\theta \cdot$$

$$\cdot \int_{4\mu^2}^{\infty} \varrho(s)\,\mathrm{d}s \left\{ \frac{1}{s - \mu^2 + 2p\omega_q - 2pq\cos\theta} + \frac{1}{s - \mu^2 + 2p\omega_q + 2pq\cos\theta} \right\}.$$

The denominators of the second and third integrals in the r.h.s. can never vanish in the physical region for channel I. The only contribution to the imaginary part of $B(q^2)$ comes from the first term

$$\operatorname{Im} B(q^2) = \varrho(4q^2 + 4\mu^2).$$

We can then replace the weight function ϱ by the imaginary part of B.

$$(10) \qquad B(\nu) = \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im} B(\nu')\,\mathrm{d}\nu'}{\nu' - \nu - i\varepsilon} - \frac{3}{4\pi}\int_{-1}^{+1} \sin^2\theta\,\mathrm{d}\cos\theta \int_0^{\infty} \operatorname{Im} B(\nu')\,\mathrm{d}\nu' \cdot$$

$$\cdot \frac{2\nu' + \nu + \frac{9}{4}}{\left[\nu' + \frac{\nu}{2} + \frac{9}{8}\right]^2 - \frac{\nu}{\nu+1}\left(\frac{\nu}{2} + \frac{3}{8}\right)^2 \cos^2\theta},$$

where the following change of variables has been made:

$$\frac{s}{4} - 1 = \nu', \qquad q^2 = \nu;$$

the pion mass μ has been taken to be unity.

We now have to apply the unitarity requirement. For this purpose we use the relation

$$\operatorname{Im} T = TT^+.$$

In terms of matrix elements this becomes:

$$\operatorname{Im} \langle \gamma\pi | T | \pi\pi \rangle = \sum_{E_n = E_{\pi\pi}} \langle \gamma\pi | T | n \rangle \langle n | T^+ | \pi\pi \rangle.$$

When s_1 is less than $(4\mu)^2$, the only intermediate state contributing is a two pion state (the $\gamma\pi$ state is disregarded because it corresponds to higher order in the electromagnetic coupling constant); this state has the same quantum number as the final two pion state: $J=1$, $T=1$. The unitarity condition takes the simple form:

$$\text{Im}\, B(\nu) = B(\nu)\, h^*(\nu)\,, \tag{11}$$

where $h(\nu) = \exp[i\delta(\nu)] \sin\delta(\nu)$ is the amplitude for the pion pion scattering in the $T=1$, $J=1$ state. When $s_1 = 4\nu+4$ is larger than 16, this relation is no longer rigorous but we maintain it because there is no way of taking into account properly the four pion states.

Equation (10) becomes a homogeneous Mushkelishvili-Omnès integral equation ([10]) for the quantity $B(\nu)$:

$$B(\nu) = \frac{1}{\pi}\int_0^\infty \frac{B(\nu')\, h^*(\nu')\, \mathrm{d}\nu'}{\nu' - \nu - i\varepsilon} + \frac{3}{4\pi}\int_{-1}^{+1} \sin^2\theta\, \mathrm{d}\cos\theta \int_0^\infty B(\nu')\, h^*(\nu')\, \mathrm{d}\nu' \cdot \frac{2\gamma' + \nu + \frac{9}{4}}{\left(\nu' + \frac{\nu}{2} + \frac{9}{8}\right)^2 - \frac{\nu}{\nu+1}\left(\frac{\nu}{2} + \frac{3}{8}\right)^2 \cos^2\theta}\,. \tag{12}$$

5. – Reduction of the integral equation to a Fredholm type equation.

We want to put equation (12) in the form:

$$B(\nu) = \frac{1}{\pi}\int_0^\infty \frac{B(\nu')\, h^*(\nu')\, \mathrm{d}\nu'}{\nu' - \nu - i\varepsilon} + \frac{1}{\pi}\int_{-\infty}^{-a} \frac{c(z)\, \mathrm{d}z}{\nu - z}\,. \tag{13}$$

Then it is straightforward to transform equation (12) into a Fredholm equation.

The problem is to determine the left hand cut and the discontinuity across this cut of the function $B(\nu)$. The main difficulty of this transformation is due to kinematical complications coming from the presence of unequal masses.

Let us study the kernel $K(\nu, \nu')$ defined as the integral over $\cos\theta$ in the second term of the right hand side of (12):

$$K(\nu, \nu') = \frac{6}{\pi}\int_0^{+1} (1-u^2)\, \frac{(\nu+1)\left(\nu + 2\nu' + \frac{9}{4}\right)}{(\nu+1)\left[\nu + 2\nu' + \frac{9}{4}\right]^2 - \nu\left(\nu + \frac{3}{4}\right)^2 u^2}\, \mathrm{d}u\,. \tag{14}$$

([10]) R. OMNÈS: *Nuovo Cimento*, **8**, 316 (1958).

It will be shown that the denominator $D(\nu, \nu', u^2)$ of the above integrand has three real negative roots with respect to ν. This enables us to define a left hand cut. These roots are functions of ν' and u^2. The integrand will be decomposed in partial fractions with respect to ν and put in the form

$$\sum_{i=1,2,3} \frac{\beta_i(\nu', u^2)}{\nu - \alpha_i(\nu', u^2)} \,. \tag{15}$$

Then α_i will be taken as a new variable of integration instead of u. It is not necessary to know the roots α_i of D but rather the expression of u in terms of α_i, and this turns out to be very simple, because D is linear in u^2. The existence and location of the roots are quite trivially established by the following table:

$$D(-1, \nu', u^2) = \frac{u^2}{16} > 0 \,,$$

$$D\left(-\frac{9}{8}, \nu', u^2\right) = -\frac{\nu'}{2}\left(\nu' + \frac{9}{8}\right) - \left(\frac{3}{8}\right)^3 (1 - u^2) < 0 \,,$$

$$D(-2\nu' - \tfrac{9}{4}, \nu', u^2) = (8\nu' + 9)(\nu' + \tfrac{3}{4})^2 u^2 > 0 \,,$$

$$D(-\infty, \nu', u^2) < 0 \,,$$

$$-\infty \leqslant \alpha_3 \leqslant -2\nu' - \tfrac{9}{4} \leqslant \alpha_2 \leqslant -\tfrac{9}{8} \leqslant \alpha_1 \leqslant -1 \,. \tag{16}$$

For given ν' it can be shown that the three roots $\alpha_i(\nu', u^2)$ are *monotonous* functions of u^2. The limiting values are:

	$u = 0$	$u = 1$
α_1	-1	$\alpha_m(\nu') \geqslant -\frac{9}{8}$
α_2	$-2\nu' - \frac{9}{4}$	$\alpha_M(\nu') \leqslant -\frac{9}{8}$
α_3	$-2\nu' - \frac{9}{4}$	$-\infty$

$\alpha_m(\nu')$ and $\alpha_M(\nu')$ are roots of $D(\nu, \nu', 1) = 0$, which in this special case becomes a second degree equation. In the limit $\nu' = 0$

$$\alpha_m(0) = \alpha_M(0) = -\tfrac{9}{8} \,.$$

This means that when u and ν' vary in the whole physical domain one obtains a single continuous cut from $-\infty$ to -1.

In each term of the decomposition (15) we take $\alpha_i = z$ as a new variable. Since for fixed ν' the ranges of variation of $\alpha_1 \alpha_2 \alpha_3$ do not overlap, the integrand

is reduced to a single term:

$$\frac{1}{2}\frac{(z+1)}{(z+\frac{3}{4})}\,\frac{z(z+\frac{3}{4})^2-(z+1)(z+2\nu'+\frac{9}{4})^2}{(z+\frac{3}{4})(z^2+(z/2)-\nu'-\frac{9}{8})-z(z+1)(z+2\nu'+\frac{9}{4})}\,.$$

We now study the Jacobian of the transformation

$$\frac{\mathrm{d}u}{\mathrm{d}z}=\frac{1}{2\sqrt{u^2}}\,\frac{\mathrm{d}(u^2)}{\mathrm{d}z}\,,$$

with

$$\sqrt{u^2}=\sqrt{\frac{z+1}{z}}\,\frac{z+2\nu'+\frac{9}{4}}{z+\frac{3}{4}}\,\varepsilon\,,$$

for $z>-2\nu'-\frac{9}{4}$, *i.e.* $z=\alpha_1$ or α_2, $\varepsilon=-1$; for $z=\alpha_3$, $\varepsilon=+1$. Therefore

$$K(\nu,\nu')=\frac{1}{\pi}\left[\int_{-\infty}^{-2\nu'-(9/4)}+\int_{-2\nu'-(9/4)}^{\alpha_M(\nu')}-\int_{\alpha_m(\nu')}^{-1}\right]\mathrm{d}z\,\frac{A(\nu',z)}{\nu-z}\,. \tag{17}$$

$A(\nu',z)$ turns out to be given by

$$A(\nu',z)=12(\nu'+1)\,\frac{[z-\alpha_M(\nu')][z-\alpha_m(\nu')]}{z(z+\frac{3}{4})^3}\sqrt{\frac{z+1}{z}}\,. \tag{18}$$

From this expression one can get equation (12) in the form (13) putting in evidence the discontinuity across the left hand cut:

$$C(z)=-\varepsilon\left(z+\frac{9}{8}\right)\int_0^{\nu'_M(z)}B(\nu')h^*(\nu')A(\nu',z)\,\mathrm{d}\nu'\,, \tag{19}$$

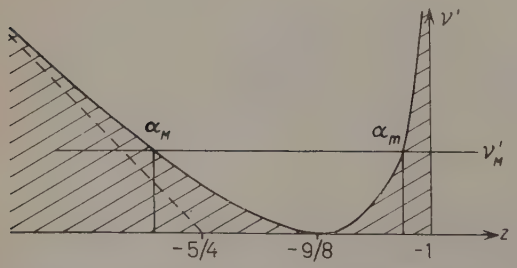

Fig. 5. The domain of integration over ν' and z.

where $\varepsilon(x)=x/|x|$ and the upper limit of integration is related to ν' by

$$\nu'_M(z)=-\frac{9}{8}-\frac{z}{2}-\frac{1}{2}(z+\tfrac{3}{4})\sqrt{\frac{z}{z+1}}\,.$$

Fig. 5 shows the domain of variation of z and ν' for physical values of ν.

The solution of equation (13) reads ([10]):

$$B(\nu) = B_H(\nu) + \frac{\exp[\varrho(\nu) + i\delta(\nu)]}{\pi} \int_{-\infty}^{-1} \frac{C(z) \exp[-\varrho(z)]\, \mathrm{d}z}{\nu - z}, \tag{20}$$

where

$$\varrho(\nu) = \frac{P}{\pi} \nu \int_0^{\infty} \frac{\delta(\nu')\, \mathrm{d}\nu'}{\nu'(\nu' - \nu)} \tag{21}$$

and $B_H(\nu)$ obeys the equation:

$$B_H(\nu) = \frac{1}{\pi} \int_0^{\infty} \frac{B_H(\nu') h^*(\nu)\, \mathrm{d}\nu}{\nu' - \nu - i\varepsilon}. \tag{22}$$

The possible solutions of (22) are

$$B_H(\nu) = \lambda \nu^m \exp[\varrho + i\delta], \tag{23}$$

where m is an algebraic integer ([10]). If one assumes for the $\pi\pi$ $J=1$ $T=1$ phase shift the behaviour $\delta(0) = 0$, $\delta(\infty) = \pi$, which corresponds to a single resonance, and furthermore that $\delta(\nu) - \pi$ goes to zero fast enough as ν goes to infinity, one sees that $\exp[\varrho(\nu)]$ behaves like $1/\nu$ at infinity. Since from (22) $B_H(\nu)$ behaves also like $1/\nu$ at infinity, the only acceptable solution is

$$B_H(\nu) = \lambda \exp[\varrho + i\delta]. \tag{23'}$$

Hence (12) is finally transformed into the following Fredholm equation

$$B(\nu) = \exp[\varrho(\nu) + i\delta(\nu)] \left[\lambda + \int_0^{\infty} N(\nu, \nu') h^*(\nu') B(\nu')\, \mathrm{d}\nu'\right], \tag{24}$$

where

$$N(\nu, \nu') = \frac{1}{\pi} \left[\int_{-\infty}^{\alpha_M(\nu')} - \int_{\alpha_m(\nu')}^{-1}\right] \mathrm{d}z\, \frac{A(\nu', z) \exp[-\varrho(z)]}{\nu - z}. \tag{25}$$

The constant λ is an arbitrary parameter which enters as a multiplying factor in the final solution. There is no way in this approach to relate this factor to other fundamental constants, in particular to the electromagnetic coupling constant which, in fact, certainly plays a fundamental role in the detailed mechanism of the interaction as can be seen by inspection of Feynman graphs. Therefore a direct confrontation of the present calculation with experiment

cannot be made. One has either to compare two measurable processes where the $\gamma+\pi\to\pi+\pi$ reaction plays a role, for instance pion photoproduction on nucleons and Compton scattering on nucleons or to use the Chew-Low extrapolation method ([11]). This method, which would give an absolute measurement of the unknown parameter can be applied to two cases:

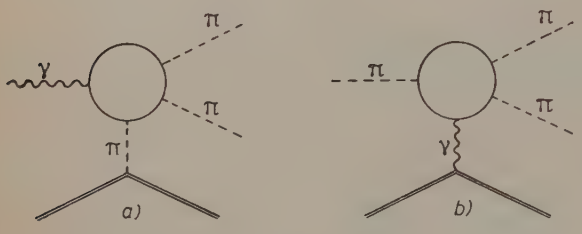

Fig. 6.

a) photoproduction of two pions on a nucleon ([12]) (Fig. 6-*a*);

b) production of a pion by a pion in the Coulomb field of a heavy nucleus (Fig. 6-*b*).

A possible way of avoiding a difficult extrapolation in process *b*) consists in using various targets and putting in evidence a Z^2 effect in the cross-section. This method, which is also suggested by B. FERRETTI for the reaction $\gamma+\pi\to\pi+\pi+\pi$, is intended to be applied by O. PICCIONI ([13]).

Another equivalent way of introducing multiplicative parameter λ is to make one subtraction in the Mandelstam representation at an arbitrary reference point. The two methods give identical results.

6. – Approximate solutions.

In this section we present approximate solutions under the assumption that the $J=1$, $T=1$ resonance is dominant in the $\pi\pi$-scattering amplitude $h(\nu)$. Since nothing is really experimentally known about this scattering amplitude we do not think it is worthwhile to work out a rigorous solution of integral equation (24) which would probably necessitate machine computation. We only hope to reproduce general features of the matrix element and to see how much the pion pion resonance is felt in this reaction. FRAZER and FULCO ([4]) have estimated the effect of this resonance on the nucleon form factor. We use a resonance energy which is approximately the same, *i.e.* $\nu_R=4$ (*).

([11]) G. F. CHEW and F. E. LOW: *Phys. Rev.*, **102**, 1635 (1959).

([12]) B. BOSCO and V. DE ALFARO: *Phys. Rev.*, **115**, 215 (1959).

([13]) B. FERRETTI: private communication.

(*) This corresponds to the quantity produced by FRAZER and FULCO in their first paper. Calculations with the new values are given in Appendix II.

6·1. *Approximation* « $\cos\theta = 0$ ». – If we go back to equation (12) and look at the denominator of the second integral in the right hand side, we see that for $\nu = \nu_R = 4$ the relative coefficient of the $\cos^2\theta$ term is at most .45; on other hand a weighting factor $\sin^2\theta$ in front of the whole integrand strongly favours $\cos^2\theta = 0$. It can be checked by analyticcal culation of the kernel $K(\nu, \nu')$ that this is not a so bad approximation. Such an approximation has been used by CINI, FUBINI and STANGHELLINI [14] in the fixed angle nucleon dispersion relations problem and gives results in good agreement with the more rigorous calculations of NOYES [15].

Equation (12) becomes

$$B_a(\nu) = \frac{1}{\pi}\int_0^\infty \frac{B_a(\nu')\,h^*(\nu')\,\mathrm{d}\nu'}{\nu' - \nu - i\varepsilon} + \frac{4}{\pi}\int_0^\infty \frac{B_a(\nu')\,h^*(\nu')\,\mathrm{d}\nu'}{2\nu' + \nu + \frac{9}{4}} \,. \tag{26}$$

It is readily transformed into the Fredholm equation

$$B_a(\nu) = \exp\left[\varrho(\nu) + i\delta(\nu)\right]\left[\lambda + \frac{4}{\pi}\int_0^\infty \frac{B_a(\nu')\,h^*(\nu')\exp\left[-\varrho(-2\nu' - \frac{9}{4})\right]\mathrm{d}\nu'}{2\nu' + \nu + \frac{9}{4}}\right] . \tag{27}$$

It is convenient to make the following change of function

$$B_a(\nu) = \exp\left[\varrho(\nu) + i\,\delta(\nu)\right]\varphi(\nu) \,;$$

then

$$\varphi(\nu) = \lambda + \frac{4}{\pi}\int_0^\infty \frac{\varphi(\nu')\sin\delta(\nu')\exp\left[\varrho(\nu') - \varrho(-2\nu' - \frac{9}{4})\right]\mathrm{d}\nu'}{2\nu' + \nu + \frac{9}{4}} \,. \tag{28}$$

This equation is easier to solve, because of more rapid convergence of the integral, in subtracted form:

$$\frac{\varphi(\nu)}{\nu + \frac{3}{4}} \equiv F(\nu) = \frac{\varphi(-\frac{3}{4})}{\nu + \frac{3}{4}}\frac{2}{\pi} - \frac{2}{\pi}\int_0^\infty \frac{\sin\delta(\nu')\exp\left[\varrho(\nu') - \varrho(-2\nu' - \frac{9}{4})\right]F(\nu')\,\mathrm{d}\nu'}{2\nu' + \nu + \frac{9}{4}} \,, \tag{29}$$

($\nu = -\frac{3}{4}$ corresponds to $s_1 = s_2 = s_3 = \mu^2$).

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[14] M. CINI, S. FUBINI and A. STANGHELLINI: *Phys. Rev.*, **114**, 1633 (1959).

[15] H. P. NOYES and D. Y. WONG: to be published; summary by G. F. CHEW in the *Proc. of the International Conference on Nuclear Forces and the Few Nucleon Problem*, 1959 (to be published, London); also private communication from H. P. NOYES to S. FUBINI.

If we first make the approximation that the width of the resonances is extremely small, equation (29) can be put in a form which has an exact solution:

$$F(\nu)=\frac{\varphi(-\frac{3}{4})}{\nu+\frac{3}{4}}-\frac{2}{\pi}\frac{\exp[-\varrho(-2\nu_R-\frac{9}{4})]}{\nu+2\nu_R+\frac{9}{4}}\int_0^\infty \sin\delta(\nu')\exp[\varrho(\nu')]F(\nu')\,\mathrm{d}\nu' , \tag{30}$$

making use of the relation

$$\frac{1}{\pi}\int_0^\infty \frac{\sin\delta(\nu)\exp[\varrho(\nu)]\,\mathrm{d}\nu}{\nu+\nu_0}=\exp[\varrho(-\nu_0)] ,$$

derived in Appendix I, we readily get the solution

$$F(\nu)=\varphi(-\tfrac{3}{4})\left[\frac{1}{\nu+\frac{3}{4}}-\frac{2}{3}\frac{\exp[\varrho(-\frac{3}{4})-\varrho(-2\nu_R-\frac{9}{4})]}{\nu+2\nu_R+\frac{9}{4}}\right] . \tag{31}$$

In the case of zero width, *i.e.* $\delta(\nu)=\pi\theta(\nu-\nu_R)$ (where $\theta(x)$ is the step function), $\exp[-\varrho(z)]$, for negative values of z, is given by

$$\exp[-\varrho(z)]=\frac{\nu_R-z}{\nu_R} . \tag{32}$$

Then $\varphi(\nu)$ becomes:

$$\varphi(\nu)=\varphi(-\tfrac{3}{4})\frac{3+8\nu_R-4\nu}{9+8\nu_R+4\nu} \tag{33}$$

and the P wave amplitude becomes

$$A(\nu)=\frac{1}{4}\sqrt{\frac{\nu(4\nu+3)}{4\nu+5}}\exp[\varrho(\nu)+i\delta(\nu)]\varphi\left(-\frac{3}{4}\right)\frac{3+8\nu_R-4\nu}{9+8\nu_R+4\nu} . \tag{33'}$$

This amplitude is dominated by the $\pi\pi$ resonance since the factor $\exp[\varrho(\nu)]$ has a very sharp maximum at $\nu=\nu_R$, but, due to the properties of $\varphi(\nu)$, it vanishes at an energy $\nu=\frac{3}{4}+2\nu_R$, *i.e.* about 8.75 for $\nu_R\sim 4$; this is a fairly large energy where the Mandelstam treatment might break down and one should not consider this feature of the solution too seriously, but we believe that an exact treatment of the integral equation will qualitatively give the same result because the approximation « $\cos\theta=0$ » slightly underestimates the kernel of the Fredholm equation.

Keeping the same analytical form (31) with undetermined coefficients we have tried to solve equation (29) for a non-zero width. We have constructed a P wave amplitude, characterized by a pole at $\nu=3.75-2i$ in the second

Riemann sheet corresponding to a resonance (see Appendix). It turns out that the trial form of the solution is very suitable. The numerical result is:

$$\varphi(\nu) = \varphi\left(-\frac{3}{4}\right)\frac{56.4 - 4\nu}{65.4 - 8\nu}.$$

This should be compared to:

i) the case of zero width which gives a factor

$$\frac{35 - 4\nu}{41 + 4\nu}.$$

ii) application of formula (31), taking the $\exp[\varrho(z)]$ corresponding to the actual width:

$$\frac{51 - 4\nu}{58 + 5.7\nu}.$$

All these solutions have the same qualitative behaviour. As could be expected formula (31) gives an intermediate result between the exact solution and the limiting case of zero width.

6·2. *Possibility of more refined calculations.* – One of the difficulties of the present calculation is that when the previous approximation is abandoned, the kernel $N(\nu, \nu')$ is given as an integral over z (Eq. (25)). For special forms of the factor $\exp[-\varrho(z)]$ the integral can be performed.

We have already given in (32) the very simple form of this factor for the case of zero width. More generally we have checked on the case of non zero width considered in Section 6·1 and in the Appendix that a linear form

$$\exp[-\varrho(z)] = 1 - z/\alpha$$

can be used for $-20 < z < 0$.

Let us compare the integrands of $K(\nu, \nu')$ and $N(\nu, \nu')$ which are then respectively

$$\frac{A(\nu', z)}{\nu - z} \quad \text{and} \quad \frac{A(\nu', z)}{\nu - z}\cdot\frac{\alpha - z}{\alpha},$$

the last one can be written

$$\lim_{\nu\to\infty}\left[\frac{\nu A(\nu', z)}{\alpha(\nu - z)}\right] + \frac{\alpha - \nu}{\alpha}\frac{A(\nu', z)}{\nu - z},$$

so that the connection between K and N is just

$$\alpha N(\nu, \nu') = \lim_{\nu \to \infty} [\nu K(\nu, \nu')] + (\alpha - \nu) K(\nu, \nu') , \tag{34}$$

K can be directly computed in the form (14):

$$K(\nu, \nu') = \frac{3}{\pi}\left[2 \frac{(\nu+1)(\nu+2\nu'+\frac{9}{4})}{\nu(\nu+\frac{3}{4})^2} - \right. \tag{35}$$

$$\left. - \sqrt{\frac{\nu+1}{\nu}} \frac{(\nu+1)(\nu+2\nu'+\frac{9}{4})^2 - \nu(\nu+\frac{3}{4})^2}{\nu(\nu+\frac{3}{4})^3} \log \frac{\sqrt{\nu+1}\,(\nu+2\nu'+\frac{9}{4}) + \sqrt{\nu}(\nu+\frac{3}{4})}{\sqrt{\nu+1}\,(\nu+2\nu'+\frac{9}{4}) - \sqrt{\nu}(\nu+\frac{3}{4})}\right] .$$

then since $\lim_{\nu \to \infty} (\nu K(\nu, \nu'))$ is independent of ν', one can write the integral equation as:

$$B(\nu) = B(\alpha) + \frac{\alpha - \nu}{\alpha} \int_0^\infty K(\nu, \nu') h^*(\nu') B(\nu') \, d\nu' . \tag{36}$$

7. – Concluding remarks.

Starting from the Mandelstam representation of the matrix element for photoproduction of pions on pions, we have derived an integral equation for the $J = 1$ reaction amplitude. Under the assumption of a sharp resonance in the $J = 1$ $T = 1$ π-π scattering, we have given explicit approximate forms of the solution which exhibit the same resonant behaviour. By inserting in the Mandelstam representation this result one could estimate the $J = 3$ state contribution and see if the approximation made in going from a two dimensional representation to a sum of unidimensional representations is justified.

We have already noticed that the solution contains an arbitrary multiplicative constant which cannot be determined in a theoretical way in the present approach. On the other hand, the validity of this treatment certainly breaks down for total c.m. energies appreciably larger than 4.

The knowledge of this amplitude and of the pion pion amplitude permits to determine explicitly, within the same framework, the Compton effect on pions (*). This amplitude is also needed in order to apply the Mandelstam treatment to photopion production on nucleons and, then, to Compton scattering on nucleons.

Our calculation also indicates a possible enhancement of the three pion contribution ([12]) to the isoscalar part of the nucleon form factors; one cannot trust very much this indication since here we are dealing with a real photon.

(*) This result will be published later.

We have indicated a way of determining the unknown factor in the solution by extrapolation of the cross-section for double photoproduction; the measuring of these cross-sections is certainly very difficult since all the kinematics must be obtained from the final products of the reaction, the energy of the photon being unknown. Another way would be to extrapolate data on single pion production by a pion in the Coulomb field of a heavy nucleus; but then the recoil is difficult to measure; it is better [13] to use both a Z^2 effect and angular correlation effects to isolate this phenomenon.

* * *

We wish to thank Professors M. CINI and S. FUBINI for many useful suggestions during the course of this work and Professor B. FERRETTI for a fruitful discussion. Mr. W. KLEIN is acknowledged for high speed hand computations.

One of us (M.G.) would like to thank Professors C. J. BAKKER and M. FIERZ for hospitality at the CERN Theoretical Studies Division and the French C.N.R.S. for financial support.

APPENDIX I

We assume a $J=1$, $T=1$ monotonous phase shift, with a single resonance, such that

$$\delta(\nu) \sim \text{const}\ \nu^{\frac{3}{2}} \qquad \text{for } \nu \to 0,$$

$$|\pi - \delta(\nu)| < \text{const}\ \nu^{-\frac{1}{2}} \qquad \text{for } \nu \to +\infty,$$

then

$$\varrho(\nu) = \nu \frac{P}{\pi} \int_0^\infty \frac{\delta(\nu')\,\mathrm{d}\nu'}{\nu'(\nu' - \nu)}$$

is such that

$$\varrho(\nu) \simeq \nu \frac{1}{\pi} \int_0^\infty \frac{\delta(\nu')\,\mathrm{d}\nu'}{\nu'^2} \qquad \text{for } \nu \to 0,$$

$$\varrho(\nu) \simeq \text{const} - \log \nu \qquad \text{for } \nu \to \infty.$$

The quantity

$$\int_0^\infty \frac{\exp[\varrho(\nu)] \sin \delta(\nu)\,\mathrm{d}\nu}{\nu + \nu_0} = \frac{1}{2i} \int_0^\infty \frac{\exp[\varrho(\nu) + i\,\delta(\nu)] - \exp[\varrho(\nu) - i\,\delta(\nu)]}{\nu + \nu_0}\,\mathrm{d}\nu$$

can be evaluated by contour integration (Fig. A1) of the function

$$\frac{1}{z+\nu_0}\exp\left[\frac{1}{\pi}\int_0^\infty \frac{\delta(\nu')\,d\nu'}{(\nu'-z)}\right];$$

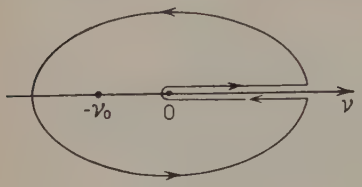

Fig. A1.

it is equal to

$$\pi \exp\left[\varrho(-\nu_0)\right].$$

In the special case of zero width resonance at $\nu = \nu_R$

$$\exp\left[\varrho(\nu)\right] = \left|\frac{\nu_R}{\nu_R - \nu}\right|$$

and then

$$\int_0^\infty \frac{\exp\left[\varrho(\nu)\right]\sin\delta(\nu)}{\nu+\nu_0}\,d\nu \to \pi\,\frac{\nu_R}{\nu_R+\nu_0}$$

and

$$\int_0^\infty \exp\left[\varrho(\nu)\right]\sin\delta(\nu)\,d\nu \to \pi\nu_R .$$

We have built a model of scattering amplitude with non-zero width of the resonance such that $\exp[\varrho]$ could be evaluated easily. It turns out that the minimum number of poles needed in the S-matrix is four if one wants to reproduce the features indicated at the beginning of the Appendix. One gets

$$\exp[i\delta]\sin\delta = \frac{[2\Gamma + K_1 - K_0][\nu]^{\frac{1}{2}}}{(K_0 + i\sqrt{\nu})(K_1 - i\sqrt{\nu})\left[(\Gamma - i\sqrt{\nu})^2 + \nu_R^2\right]}.$$

With the conditions

$$2\Gamma K_0 K_1 = (K_1 - K_0)(\Gamma^2 + \nu_R),$$
$$K_1 > K_0 > 0.$$

The only pole of $\exp[i\delta]\sin\delta$ in the *first* Riemann sheet is $\nu = -K_0^2$, it has no zeros except at $\nu = 0$. Therefore the quantity

$$(z)^{\frac{1}{2}}\exp\left[\frac{z}{\pi}\int_0^\infty \frac{\delta(\nu)\,d\nu}{\nu(\nu-z)}\right]\Big/\left[z+K_0^2\right]\exp\left[i\delta(z)\right]\sin\delta(z)$$

is regular in the whole z plane, real along the cut $0 - \infty$, and has a finite non zero limit as $|z| \to \infty$. It is therefore a real constant.

Finally

$$\exp[\varrho(\nu)] = \frac{K_1}{K_0}(\Gamma^2 + \nu_R)\sqrt{\frac{K_0^2 + \nu}{(K_1^2 + \nu)[(\Gamma^2 + \nu_R - \nu)^2 + 4\nu\Gamma^2]}} \qquad \text{for } \nu > 0,$$

$$\exp[\varrho(\nu)] = \frac{K_1}{K_0}(\Gamma^2 + \nu_R)\frac{K_0 + \sqrt{-\nu}}{(K_1 + \sqrt{-\nu})(\Gamma^2 + \nu_R - \nu + 2\Gamma\sqrt{-\nu})} \qquad \text{for } \nu < 0.$$

The parameters used in Section 6·1 are:

$$\nu_R = 4\,, \qquad \Gamma = 0.5\,, \qquad K_0 = 1.755\,, \qquad K_1 = 2.995\,.$$

This corresponds approximately to the case

$$\Gamma = 0.3 \qquad \nu_R = 3.5$$

in Frazer and Fulco's notations [4].

Appendix II

With $\nu_R = 1.7375$, $\Gamma = 0.25$, $K_0 = 1.12$, $K_1 = 1.62$, corresponding to the new data of Frazer and Fulco [4], we get:

$$\varphi = \frac{34 - 4.34\nu}{31.5 + 5.66\nu}\,.$$

This solution is extremely close to the approximate solution obtained form eq. (31). On the other hand the approximation « $\cos\theta = 0$ » is much better justified in the present case.

RIASSUNTO (*)

Si calcola la fotoproduzione di pioni su pioni per mezzo della versione di Cini-Fubini della tecnica di Mandelstam. Il problema è ridotto alla risoluzione di un'equazione integrale di Fredholm contenente una costante moltiplicatrice arbitraria che potrebbe essere determinata per estrapolazione delle sezioni trasversali sperimentali. Col presupposto di una netta risonanza pione-pione deriviamo soluzioni approssimate che presentano lo stesso comportamento di risonanza. Questi risultati saranno fra poco applicati allo scattering Compton su pioni e nucleoni e alla fotoproduzione su nucleoni.

(*) *Traduzione a cura della Redazione.*

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Uniqueness of the Orbital Angular Momentum Operators.

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(ricevuto il 30 Dicembre 1959)

Summary. — In a separable Hilbert space in which the three Cartesian coordinates form a complete set of commuting variables, it is proved that Hermitian operators which satisfy the commutation relations for the angular momentum operators are unique within a unitary transformation.

1. – Introduction.

The question of the uniqueness of quantum mechanical operators is a problem of long standing interest. Dirac has shown (1) the uniqueness of the linear momentum operators in the following sense. Let L_2 be the Hilbert space of all square integrable complex functions of three real variables x_1, x_2, x_3, where $-\infty < x_1,\ x_2,\ x_3 < \infty$. Let $\boldsymbol{q} = (q_1, q_2, q_3)$ and $\boldsymbol{p} = (p_1, p_2, p_3)$ be self-

(*) The work of H. E. M. is sponsored by the Office of Ordnance Research, U.S. Army, Contract DA-30-069-ORD-2154. Reproduction in whole or in part permitted for any purpose of the U. S. Government.

(1) P. A. M. Dirac: *The Principles of Quantum Mechanics*, 4th ed., Chap. 22 (Oxford, 1958), p. 89. See also: H. Weyl: *Gruppentheorie und Quantenmechanik* (Berlin, 1928), pp. 83, 207; M. H. Stone: *Proc. Nat. Acad. Sci.*, **16**, 172 (1930); J. Von Neumann: *Math. Ann.*, **104**, 570 (1931); F. Rellich: *Gött. Nachr.*, 107 (1946); G. W. Mackey: *Duke Math. Journ.*, **16**, 313 (1949).

adjoint vector operators in L_2 defined by

$$\boldsymbol{q}\psi(\boldsymbol{x}) = \boldsymbol{x}\psi(\boldsymbol{x}), \qquad \psi \in L_2, \tag{1.1}$$

$$\boldsymbol{p}\psi(\boldsymbol{x}) = -i\boldsymbol{\nabla}\psi(\boldsymbol{x}). \tag{1.2}$$

Then

$$[q_j, p_k] = i\delta_{jk} \qquad j, k = 1, 2, 3. \tag{1.3}$$

Let $\bar{\boldsymbol{p}}$ be any any other self-adjoint vector operator in L_2 satisfying

$$\begin{cases} [q_j, \bar{p}_k] = i\delta_{jk}, \\ [\bar{p}_j, \bar{p}_k] = 0. \end{cases} \tag{1.4}$$

Then there exists a unitary operator U which commutes with q and which transforms $\bar{\boldsymbol{p}}$ into $\boldsymbol{p}$

$$U\bar{\boldsymbol{p}}U^{-1} = \boldsymbol{p}. \tag{1.5}$$

The proof of the preceding statement depends heavily on the following standard theorem of vector analysis which can easily be proved by a Fourier expansion. Every vector field $\boldsymbol{f}(\boldsymbol{x})$ satisfying the relation $\boldsymbol{p}\times\boldsymbol{f}(\boldsymbol{x}) = 0$ has the form $\boldsymbol{f}(x) = \boldsymbol{p}\varphi(x)$, where $\varphi(\boldsymbol{x})$ is a scalar field. This article will be devoted to proving a result for the orbital angular momentum operators which is analogous to Dirac's result for the linear momentum operator $\boldsymbol{p}$. The proof will depend heavily on a result analogous to the standard theorem of vector analysis stated above.

2. – The Theorem.

We shall retain the notation introduced above and define the orbital angular momentum operators $\boldsymbol{L} = (L_1, L_2, L_3)$ by

$$\boldsymbol{L} = \boldsymbol{q}\times\boldsymbol{p}. \tag{2.1}$$

These operators satisfy the commutation relations

$$[q_j, L_j) = 0, \qquad (j = 1, 2, 3) \tag{2.2}$$

$$[q_1, L_2] = -[q_2, L_1] = iq_3 \quad \text{(cycl.)}, \tag{2.3}$$

$$\boldsymbol{L}\times\boldsymbol{L} = i\boldsymbol{L}. \tag{2.4}$$

We shall show that if $\overline{\boldsymbol{L}}$ is any other self-adjoint operator satisfying

$$[q_j, \overline{L}_j] = 0\,, \qquad (j=1,2,3) \tag{2.5}$$

$$[q_1, \overline{L}_2] = -[q_2, \overline{L}_1] = iq_3 \quad \text{(cycl.)}, \tag{2.6}$$

$$\overline{\boldsymbol{L}}\times\overline{\boldsymbol{L}} = i\overline{\boldsymbol{L}}\,, \tag{2.7}$$

then there is a unitary operator U which commutes with $\boldsymbol{q}$ and which transforms $\overline{\boldsymbol{L}}$ into $\boldsymbol{L}$

$$U\overline{\boldsymbol{L}}U^{-1} = \boldsymbol{L}\,. \tag{2.8}$$

To show this we note first that

$$[q_j, \overline{\boldsymbol{L}} - \boldsymbol{L}] = \boldsymbol{0} \qquad (j=1,2,3) \tag{2.9}$$

so that $\overline{\boldsymbol{L}} - \boldsymbol{L}$ is a function of $\boldsymbol{q}$,

$$\overline{\boldsymbol{L}} - \boldsymbol{L} = \boldsymbol{f}(\boldsymbol{q})\,, \tag{2.10}$$

since q_1, q_2, q_3, form a complete set of commuting variables. Substituting $\overline{\boldsymbol{L}} = \boldsymbol{L} + \boldsymbol{f}(\boldsymbol{q})$ in (2.7) we find that $\boldsymbol{f}$ must satisfy the condition

$$[L_2, f_3(\boldsymbol{q})] - [L_3, f_2(\boldsymbol{q})] = if_1(\boldsymbol{q})\,, \qquad \text{(cycl.)} \tag{2.11}$$

which can also be written as a set of first order partial differential equations

$$\boldsymbol{L}(\boldsymbol{x})\times\boldsymbol{f}(\boldsymbol{x}) = i\boldsymbol{f}(\boldsymbol{x})\,, \tag{2.12}$$

where $\boldsymbol{L}(\boldsymbol{x}) = -i\boldsymbol{x}\times\boldsymbol{\nabla}$. But, as is shown in the Appendix, every $\boldsymbol{f}(\boldsymbol{x})$ satisfying (2.12) must have the form

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x})\,\varphi(\boldsymbol{x}), \tag{2.13}$$

where $\varphi(\boldsymbol{x})$ is a scalar field [2]. Equation (2.13) is the analogue of the standard theorem of vector analysis quoted earlier which was used by DIRAC in proving the uniqueness of the linear momentum operators. From (2.13) we see that

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{q})\,\psi(\boldsymbol{x}) &= (\boldsymbol{L}(\boldsymbol{x})\varphi(\boldsymbol{x}))\,\psi(\boldsymbol{x}) \\ &= [\boldsymbol{L}(\boldsymbol{x}),\, \varphi(\boldsymbol{x})]\,\psi(\boldsymbol{x}) \\ &= [\boldsymbol{L},\, \varphi(\boldsymbol{q})]\,\psi(\boldsymbol{x}) \end{aligned} \tag{2.14}$$

[2] The fact that every vector field of the form (2.13) satisfies (2.12) follows trivially from (2.4).

so that

$$\boldsymbol{f}(\boldsymbol{q}) = [\boldsymbol{L}, \varphi(\boldsymbol{q})] \tag{2.15}$$

and

$$\overline{\boldsymbol{L}} = \boldsymbol{L} + [\boldsymbol{L}, \varphi(\boldsymbol{q})] . \tag{2.16}$$

We note that

$$\big[[\boldsymbol{L}, \varphi(\boldsymbol{q})], \varphi(\boldsymbol{q})\big] = 0 , \tag{2.17}$$

since any two functions of $\boldsymbol{q}$ commute. Hence

$$\begin{aligned} \exp[-\varphi(\boldsymbol{q})]\, \boldsymbol{L} \exp[\varphi(\boldsymbol{q})] &= \boldsymbol{L} + [\boldsymbol{L}, \varphi(\boldsymbol{q})] \\ &= \overline{\boldsymbol{L}} . \end{aligned} \tag{2.18}$$

If $\boldsymbol{L}$ is to be self-adjoint, then $\boldsymbol{f}(\boldsymbol{q})$ must self-adjoint. If we write $\varphi(\boldsymbol{q}) = \chi(\boldsymbol{q}) + i\eta(\boldsymbol{q})$, where χ and η are self-adjoint, then $\boldsymbol{f}^{+}(\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{q})$ implies $[\boldsymbol{L}, \chi(\boldsymbol{q})] = 0$ and $i[\boldsymbol{L}, \eta(\boldsymbol{q})] = \boldsymbol{f}(q)$. Hence we have

$$\exp[-i\eta(\boldsymbol{q})]\, \boldsymbol{L} \exp[i\eta(\boldsymbol{q})] = \overline{\boldsymbol{L}} \tag{2.19}$$

so we can choose

$$U = \exp[i\eta(q)] . \tag{2.20}$$

3. – Relation between $\overline{\boldsymbol{L}}$ and $\overline{\boldsymbol{p}}$.

DIRAC [1] showed that

$$\overline{\boldsymbol{p}} = \boldsymbol{p} + [\boldsymbol{p}, \zeta(\boldsymbol{q})] \tag{3.1}$$

so we see that

$$\boldsymbol{q} \times \overline{\boldsymbol{p}} = \boldsymbol{L} + [\boldsymbol{L}, \zeta(\boldsymbol{q})] , \tag{3.2}$$

which is the $\boldsymbol{L}$ obtained by putting $\varphi = \zeta$. Hence the arbitrariness in the choice of orbital angular momentum operators arises from the arbitrariness in the choice of linear momentum operators.

APPENDIX

It will be shown here that every solution of the equation

$$\boldsymbol{L}(\boldsymbol{x}) \times \boldsymbol{f}(\boldsymbol{x}) = i\boldsymbol{f}(\boldsymbol{x}) \tag{A.1}$$

has the form

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x})\, \varphi(\boldsymbol{x}) . \tag{A.2}$$

To prove this we shall use a spherical harmonic expansion instead of a Fourier expansion as for the linear momentum case,

$$\boldsymbol{f}(\boldsymbol{x}) = \sum_{l,m} \boldsymbol{g}^{lm}(r)\, Y_{lm}(\theta, \varphi)\,, \qquad (\boldsymbol{g}^{lm} = g_1^{lm},\ g_2^{lm},\ g_3^{lm})\,. \tag{A.3}$$

We shall now transcribe the analytic problem into an algebraic problem. Using

$$\boldsymbol{L}(\boldsymbol{x}) Y_{lm} = \sum_{m'} \boldsymbol{L}^{(l)}_{m'm} Y_{lm'}\,, \tag{A.4}$$

where $\boldsymbol{L}^{(l)}$ is the $(2l+1)$-dimensional matrix representing $\boldsymbol{L}(\boldsymbol{x})$ in the irreducible representation Γ_l, and letting

$$g_j^{(l)} = \begin{pmatrix} g_j^{l,l} \\ g_j^{l,l-1} \\ \vdots \\ g_j^{l,-l} \end{pmatrix} \tag{A.5}$$

and

$$g^{(l)} = \begin{pmatrix} g_1^{(l)} \\ g_2^{(l)} \\ g_3^{(l)} \end{pmatrix} \tag{A.6}$$

we can write (A.1) in the matrix form

$$M^{(l)} g^{(l)} = g^{(l)}\,, \tag{A.7}$$

where

$$M^{(l)} = \begin{pmatrix} 0 & iL_3^{(l)} & -iL_2^{(l)} \\ -iL_3^{(l)} & 0 & iL_1^{(l)} \\ iL_2^{(l)} & -iL_1^{(l)} & 0 \end{pmatrix} \tag{A.8}$$

is a Hermitian matrix if $\boldsymbol{L}^{(l)}$ is chosen Hermitian.

If we let

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.9}$$

then

$$[\boldsymbol{J} \times \boldsymbol{J}] = i\boldsymbol{J}\,, \tag{A.9a}$$

$$M^{(l)} = -L_1^{(l)} \otimes J_1 - L_2^{(l)} \otimes J_2 - L_3^{(l)} \otimes J_3\,. \tag{A.10}$$

If we further let

$$\mathscr{L}^{(l)} = \boldsymbol{L}^{(l)} \otimes I \,, \tag{A.11}$$

$$\mathscr{J} = I \otimes \boldsymbol{J} \,, \tag{A.12}$$

where the I's in (A.11) and (A.12) are unit matrices of dimension 3 and $(2l+1)$ respectively, then we can write

$$M^{(l)} = -\mathscr{L}^{(l)} \cdot \mathscr{J} \,. \tag{A.13}$$

We now have $M^{(l)}$ in the desired from, and can easily determine its eigenvalues together with the degeneracy of each. First we note that

$$M^{(0)} = 0 \,, \tag{A.14}$$

and subsequently consider only $l > 0$. From here on we shall drop the superscript l. We note that $\mathscr{L} - \mathscr{J}$ is the set of infinitesimal operators of the representation

$$\Gamma_l \otimes \Gamma_1 = \Gamma_{l+1} \oplus \Gamma_l \oplus \Gamma_{l-1} \tag{A.15}$$

of the three-dimensional rotation group. Since

$$[M, \mathscr{L} + \mathscr{J}] = 0 \tag{A.16}$$

it follows from Schur's lemma that when $\mathscr{L} + \mathscr{J}$ is completely reduced, M will be diagonal with the scalar blocks on the diagonal corresponding to the three irreducible components,

$$\Gamma_{l+1}, \quad \Gamma_l, \quad \Gamma_{l-1} \,.$$

Thus the corresponding three eigenvalues will have the degeneracies $2l+3$, $2l+1$, $2l-1$. Since $(\mathscr{L}+\mathscr{J})^2 = \mathscr{L}^2 + \mathscr{J}^2 + 2\mathscr{L}\cdot\mathscr{J}$

$$M = \tfrac{1}{2}\{l(l+1)I + 2I - (\mathscr{L} + \mathscr{J})^2\} \,, \tag{A.17}$$

where I is now the identity in the product space. Since $(\mathscr{L}+\mathscr{J})^2$ is a Casimir operator (the total angular momentum), its three eigenvalues corresponding to the three irreducible components Γ_{l+1}, Γ_l, Γ_{l-1} are $(l+1)(l+2)$, $l(l+1)$, $(l-1)l$. Hence the corresponding three eigenvalues of M are $-l$, 1, $l+1$. Thus we have the following table:

Irreducible Components	Eigenvalues of M	Degeneracies
Γ_{l+1}	$-l$	$2l+3$
Γ_l	1	$2l+1$
Γ_{l-1}	$l+1$	$2l-1$

So we see that (A.7) has $(2l+1)$ linearly independent solutions for each value of l.

If we now expand φ,

$$\varphi(\boldsymbol{x}) = \sum_{l,m} v^{lm}(r) Y_{lm} \tag{A.18}$$

and let

$$v^{(l)} = \begin{pmatrix} v^{l,l} \\ v^{l,l-1} \\ \vdots \\ v^{l,-l} \end{pmatrix}, \tag{A.19}$$

$$\tilde{v}^{(l)} = \begin{pmatrix} v^{(l)} \\ v^{(l)} \\ v^{(l)} \end{pmatrix}, \tag{A.20}$$

$$\Lambda^{(l)} = \begin{pmatrix} L_1^{(l)} & 0 & 0 \\ 0 & L_2^{(l)} & 0 \\ 0 & 0 & L_3^{(l)} \end{pmatrix}, \tag{A.20}$$

We can write (A.2) in the form (dropping l superscripts)

$$\Lambda \tilde{v} = g\,. \tag{A.21}$$

If there is a solution $\tilde{v}$ of (A.21) for every g satisfying (A.6), then there is a solution φ of (A.2) for every f satisfying (A.1). We shall now show that this is so.

Let

$$\Lambda' = \begin{pmatrix} L_1 & L_1 & L_1 \\ L_2 & L_2 & L_2 \\ L_3 & L_3 & L_3 \end{pmatrix}. \tag{A.22}$$

Then (A.21) can be written

$$\tfrac{1}{3}\Lambda' \tilde{v} = g\,. \tag{A.23}$$

By using the commutation rules (2.4) we find that

$$M\Lambda' = \Lambda'\,. \tag{A.24}$$

If we denote the first $(2l+1)$ columns of Λ' by $\lambda_1, \ldots, \lambda_{2l+1}$, we see from (A.24) that

$$M\lambda_i = \lambda_i \qquad i = 1, 2, \ldots, 2l+1. \tag{A.25}$$

By choosing L_3 diagonal it can easily be seen that the $2l+1$ vectors $\lambda_1, \ldots, \lambda_{2l+1}$ are linearly independent. Hence, by (A.25) the λ's are also a complete linearly independent set of g's. Thus we can write

$$\Lambda' = (g_1, g_2, \ldots, g_{2l+1}, g_1, g_2, \ldots, g_{2l+1}, g_1, g_2, \ldots, g_{2l+1}) \,. \tag{A.26}$$

Let us now expand g in the above linearly independent set $g_1, \ldots, g_{2l+1}$,

$$g = \sum_{i=1}^{2l+1} a_i g_i \tag{A.27}$$

if we let

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2l+1} \end{pmatrix}, \tag{A.28}$$

$$\tilde{a} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}, \tag{A.29}$$

we can write (A.27) in the form

$$g = \tfrac{1}{3} \Lambda' \tilde{a} \,. \tag{A.30}$$

Combining (A.30) with (A.23) we have

$$\Lambda' \tilde{v} = \Lambda' \tilde{a} \,. \tag{A.31}$$

Hence for every g satisfying (A.7) there is a $\tilde{v}$ satisfying (A.21) which is given by

$$\tilde{v} = \tilde{a} \,. \tag{A.32}$$

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RIASSUNTO (*)

Si dimostra che, in uno spazio hilbertiano separabile in cui le tre coordinate cartesiane formano un gruppo completo di variabili commutabili, gli operatori hermitiani che soddisfano le relazioni di commutazione per gli operatori del momento angolare, sono unici in una trasformazione unitaria.

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A Free Nucleon Theory.

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Summary. — A theory of the free nucleon quantum field based on the spin $\frac{1}{2}$ wave equation proposed by Murai is given. Notable features are: 1) the nucleon field is an (8-dimensional) *irreducible* representation of its fundamental group, the 15-parameter group of all angle-preserving transformations of space-time; 2) in spite of non-vanishing mass this nucleon possesses a new quantum number β_7', the *invariant* handedness; neutron and proton can be distinguished by $\beta_7' = -1$ and $+1$, and the charge operator can be introduced as the generator of the symmetry group $\Psi' = \exp[i\alpha(\beta_7'+1)/2]\Psi$; 3) the bare nucleon necessarily has a mass spectrum; mass is conjugate to another measurable length λ associated with a nucleon, and states can be formed in which the unsharpness of the bare mass and of λ vary in a complementary way. The relevance of this new concept to the physical process of measuring the mass via interaction with other fields and to the renormalization of this nucleon field is discussed.

1. – Introduction.

In this paper it is shown how a successful theory of the free nucleon (or Ξ hyperon) quantum field can be built within the framework of the fundamental group M, 15 parameter group of all angle-preserving transformations of space-time. No other assumptions are needed than that the nucleon be a spin $\frac{1}{2}$ irreducible representation of this group. The starting point will be the spin $\frac{1}{2}$ elementary particle wave equation proposed by Y. Murai [1]. Murai had the brilliantly simple idea to derive elementary particle wave equations simply by writing the eigenvalue equations of the invariants of the generator

[1] Y. Murai: *Nucl. Phys.*, **6**, 489 (1958).

algebra, which directly express this irreducibility (for the proper subgroup). Thus if $M_{\mu\nu} = \overset{0}{M}_{\mu\nu} + S_{\mu\nu}$ are the generators of the group M (composed of orbital and spin parts, respectively) on the proposed irreducible field ψ, then the eigenvalue equations

$$Q_{0p}\psi = Q\psi\,, \qquad R_{0p}\psi = R\psi\,, \qquad W_{0p}\psi = W\psi\,, \tag{1.1}$$

where Q_{0p}, R_{0p}, W_{0p} (algebraic expressions in the $M_{\mu\nu}$) are the three invariants of the generator algebra, must furnish the wave equation plus perhaps some extra relevant information on the quantum numbers Q, R, W. For the particular case of the proposed nucleon, the middle equation furnishes a first order wave equation, and the other two reduce to algebraic constraints between the eigenvalues Q, R, W which can only be satisfied by a *small* set of irreducible representations (apparently only four, with $R = \pm 9/8,\ \pm 27/8$). This limitation of the possible irreducible representations which are allowed for a particle of a given generic type to a discrete (in fact finite and small) set is one of the attractive features of this fundamental group. Contrast it with the Dirac electron, for which the inhomogeneous Lorentz group L_{inh} allows ∞^1 admissible irreducible representations ($0 < m^2 < \infty$). This qualitative difference in the structure is due of course to the fact that M is a pure rotation group, thus *simple*, while L_{inh} is not (the subgroup of space-time translations is normal in L_{inh}).

A striking feature of this nucleon (and in fact, of any fermion field built within M) is that it has a new quantum number $\beta_7' = \pm 1$, the *invariant handedness*, which is a good quantum number in spite of the non-vanishing mass. Neutron and proton can be distinguished by their opposite (invariant) handedness. The charge operator can be introduced as the generator of one of the pair of one-parameter symmetry groups

$$\psi \to \exp\,[i\alpha\Omega_{\pm}]\psi\,, \qquad \Omega_{\pm} \equiv \frac{\pm\beta_7 + 1}{2}\,. \tag{1.2}$$

Similarly the baryon number is obtained as the generator of the phase transformations $\psi \to \exp\,[i\alpha]\psi$. Thus the rôle of the nucleon isotopic spin t_z is here played by $(\frac{1}{2})\beta_7$. An analogous way of introducing the charge in nucleon theories built up more or less *ad hoc* as pairs of Dirac particles has recently been suggested [2].

Our aim in this paper is to show how the nucleon quantum field is to be built, using the c-number wave functions which satisfy the Murai wave equa-

[2] F. GÜRSEY: *Nuovo Cimento*, **7**, 411 (1958); V. AMAR and M. PAURI: *Nuovo Cimento*, **13**, 1290 (1959).

tion. One cannot simply use the usual formalism because here we deal with the angle, not the metric, geometry of space-time, hence invariants must be formed with respect to the bigger group M, not just its subgroup L_{inh}, integrals in the 6-dimensional space must be over certain angular regions, and so on. For instance, the basic *angular* Gauss's Theorem allows the generators of the unitary transformations on the state Hilbert space induced by the total symmetry group of the theory to be expressed as certain integrals over the surfaces f:time = const. via an action principle. These are the conserved quantities of the field: the generators of space-time translation and rotations are the total 4-momentum and angular momentum, respectively, while the generator of one of the groups (1.2) is the total charge. We thus arrive at a theory reproducing all the features of free nucleons, but what is more, *a*) this nucleon is an *irreducible* representation of its fundamental group; and *b*) this fundamental group refers *only* to transformations in space-time.

This theory forces a reappraisal of mass. One can, if he desires, restrict the present theory of a *free* nucleon to a single mass, which can be identified with the conventional « bare nucleon mass ». But it is possible, by using the complete set of nucleon modes, to form states in which the nucleon mass is in principle not sharp. Formally this results from the presence of a fundamental length λ which, in a nucleon theory, must represent some measurable length associated with a nucleon. The mass M and λ are conjugate, so that in states for which M is more or less sharp, λ is more or less indeterminate, and viceversa. Section **3** is devoted to a discussion of this idea.

2. – The free nucleon quantum field.

The equations (1.1) are equivalent to the first order wave equation

$$R_{0l}\psi(X) = R\psi(X)\,, \qquad R_{0p} = \tfrac{9}{4}\beta_7(S^{\mu\nu}\overset{0}{M}_{\mu\nu} + \tfrac{5}{2})$$

written in terms of the six homogeneous co-ordinates X^μ ($\mu = 0, 1, ..., 5$) [3] and the algebraic constraints

$$Q = (\tfrac{1}{9}R)^2 - \tfrac{5}{2}\,, \qquad W = \tfrac{3}{2}Q + \tfrac{45}{16}\,. \tag{2.2}$$

[3] Greek indices will always go from 0 to 5 and refer to the six homogeneous coordinates X^μ. Middle latin letters $l, m, n, p, ...$ go from 1 to 4 and are space-time indices. The space-time metric g_{mn} is chosen $g_{11} = g_{22}$, $g_{33} = -g_{44} = 1$, $g_{mn} = 0$ $(m \neq n)$. The nonprincipal axes form for the coordinates X^0, X^5 in (2.3) is chosen advisedly for mathematical convenience. Thus for any 6-quantity A^μ, $A_i = A^i$ $(i = 1, 2, 3)$, $A_4 = -A^4$, $A_0 = -A^5$, $A_5 = -A^0$. The practice, frequent in the literature, of ignoring the difference between contravariant and covariant indices will in this case result in instant and total confusion.

The latter equations give $R = \pm\frac{9}{8}$ or $\pm\frac{27}{8}$ (1). Here the group M is the orthogonal group preserving the fundamental form

$$g_{\mu\nu}X^\mu X^\nu \equiv g_{mn}X^m X^n - 2X^0 X^5 , \tag{2.3}$$

where $(m, n = 1, ..., 4)$ (3). We are concerned with six 8×8 matrices β_μ satisfying the anticommutations law

$$\beta_\mu\beta_\nu + \beta_\nu\beta_\mu = 2g_{\mu\nu}\mathbf{1} , \tag{2.4}$$

with $\beta_7 = i\beta_{[0}\beta_1 ... \beta_{5]} = i\beta_{[0}\beta_5]\beta_1\beta_2\beta_3\beta_4$ which satisfies $\{\beta_7, \beta_\mu\} = 0$, $\beta_7^2 = 1$. $S_{\mu\nu}$ and $\overset{0}{M}_{\mu\nu}$ are the spin and orbital parts, respectively

$$\overset{0}{M}_{\mu\nu} \equiv \frac{1}{i}\left(X_\mu\frac{\partial}{\partial X^\nu} - X_\nu\frac{\partial}{\partial X^\mu}\right), \qquad S_{\mu\nu} \equiv \frac{1}{2i}\beta_{[\mu}\beta_{\nu]} , \tag{2.5}$$

of the total generator

$$M_{\mu\nu} = \overset{0}{M}_{\mu\nu} + S_{\mu\nu} \tag{2.6}$$

for the nucleon field $\psi(X)$. Note at once that since $S_{\mu\nu}$ and thus $R_{0\nu}$ are of *even* degree in the β_μ, the theory admits the symmetry $\psi \to \beta_7\psi$. One introduces the inhomogeneous co-ordinate x (4) and λ via (5)

$$x^m = \frac{X^m}{X^0} , \qquad \frac{1}{2}(x^2 + \lambda^2) = \frac{X^5}{X^0} \qquad \text{(whence } g_{\mu\nu}X^\mu X^\nu = -(X^0)^2\lambda^2\text{)}. \tag{2.7}$$

(4) x will often stand for x^m in what follows. Also, we reserve the inner product notation x^2, px, etc., exclusively for 4-inner products: $x^2 = g_{mn}x^m x^n$, etc.

(5) The dimensions of the X^μ are $[X^0] = L^{-1}$, $[X^m] = 1$, $[X^5] = L$. This is because

$$X'^0 = \mu^{-1}X^0, \; X'^m = X^m, \; X'^5 = \mu X^5 , \tag{*}$$

is the 6-rotation induced by the dilatation (change of the length unit) $x'^m = \mu x^m$, $\lambda' = \mu\lambda$; $\mu = \text{const} > 0$. It is superfluous to insert l_0 and l_0^2 factors (l_0 an arbitrary length) into the right members of (2.7) as is done in Ingraham, ref. (11), or Murai, ref. (1), in an attempt to make the X^μ « dimensionless ». For since the subgroup (*) is in no way deleted, the X^μ continue *ipso facto* to have the dimensions given above. Another way of seeing this is that with the length factors inserted, only the combinations x^m/l_0 and λ/l_0 ever occur in the theory, *i.e.*, just the coordinates x^m and λ defined by (2.7). Apropos of this question of forming dimensionless quantities: any 6-tensorial form-invariant is automatically dimensionless (in spite of the mixed dimensions of the X^μ!) simply because it is invariant-in-form under all M-rotations by definition, and changes of the unit of length are included in M as the subgroup (*).

If we assume that $\psi(X)$ is homogeneous of degree *zero* in X^μ, *i.e.*, functions of x and λ alone, solutions of (2.1) can be obtained in the form ( )

$$\psi(x,\lambda) = U(x)\left(f_+(\lambda)\,\Omega_+ + f_-(\lambda)\,\Omega_-\right)\omega(x)\,, \tag{2.8}$$

where

$$U(x) = 1 - \tfrac{1}{2}\beta x\beta_5\,, \qquad U^{-1}(x) = 1 + \tfrac{1}{2}\beta x\beta_5 \tag{2.9}$$

is a space-time dependent spinor transformation of great importance in attributing correct momentum and energy to the field (Section 4); $f_\pm(\lambda)$ are purely λ-dependent operators defined by

$$f_\pm(\lambda) \equiv \lambda^2\left\{J_{\nu_\mp}(\varkappa\lambda)Q_1 + (-1)^{\nu_\pm}\lambda J_{\nu_\pm}(\varkappa\lambda)Q_2\right\}, \tag{2.10}$$

$$\nu_\pm \equiv \pm\tfrac{4}{9}R + \tfrac{1}{2}\,, \qquad \varkappa = \text{inverse Compton wave length of the particle,}$$

where Q_1 and Q_2 are the orthogonal projections

$$Q_1 \equiv -\tfrac{1}{2}\beta_0\beta_5\,, \qquad Q_2 \equiv -\tfrac{1}{2}\beta_5\beta_0\,, \tag{2.11}$$

$\Omega_\pm$ are the orthogonal projections

$$\Omega_\pm \equiv \frac{\pm\beta_7 + 1}{2} \tag{2.12}$$

and finally $\omega(x)$ is any spinor solution of the purely space-time dependent Dirac-type wave equation

$$\left(\beta\frac{\partial}{\partial x} + M\right)\omega(x) = 0\,, \qquad M \equiv \varkappa\left(\beta_0 + \frac{1}{2}\beta_5\right). \tag{2.13}$$

The mass operator M has the properties

$$\{\beta_m, M\} = 0\,, \qquad M^2 = -\varkappa^2\,. \tag{2.14}$$

Note that this wave equation admits the good quantum number $\beta_7' = \pm 1$ for $\omega(x)$.

To solve (2.13) write $\omega(x) = w(p)\exp[ipx]$ and iterate the operator $i\beta p - M$,

(6) Murai, *loc. cit.*, pp. 497-499. Murai's β_6 and co-ordinate r are our β_0 and co-ordinate λ respectively. Set $l_0 \to 1$.

using (2.14). We get eight elementary wave functions $u_r^a(p)$, $v_r^a(p)$ defined by

$$(2.15)\quad \begin{cases} (i\not{p}+M)u(p)=0 & (-i\not{p}+M)v(p)=0\,, & p^4\equiv\sqrt{\boldsymbol{p}^2+\varkappa^2}>0\,,\\ \beta_7 u^a=au^a\,, & \beta_7 v^a=av^a\,, & (a=\pm 1),\\ s_\parallel(p)u_r=-\dfrac{r}{2}u_r\,, & s_\parallel(p)v_r=\dfrac{r}{2}v_r & (r=\pm 1),\\ s_\parallel(p)\equiv\dfrac{\boldsymbol{s}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\,, & s^1=\dfrac{1}{2i}\beta_2\beta_3 \text{ and cyclic}\,. \end{cases}$$

For each mode $\boldsymbol{p}$ there are the usual quantum numbers of the sign of the energy and the longitudinal spin, and in addition the new quantum number β', the invariant handedness. These solutions are constructed conveniently in terms of the three mutually commuting pairs of orthogonal projections

$$(2.16)\quad \Lambda_\pm(p)\equiv\frac{\pm i\beta p M+\varkappa^2}{2\varkappa^2}\,,\quad \Sigma_\pm(p)\equiv\pm s_\parallel(p)+\frac{1}{2}\,,\quad \Omega_\pm\equiv\frac{\pm\beta_7+1}{2}\,.$$

The unconventional (compared to electron theory) form of the projection $\Lambda_\pm$, *quadratic* in the mass, comes from the circumstance that here βp and M anticommute rather than commute. The unnormalized elementary solutions are then

$$(2.17)\qquad u_r^a(p)=\Lambda_-(p)\Sigma_{-r}(p)\Omega_a x\,,\qquad v_r^a(p)=\Lambda_+(p)\Sigma_r(p)\Omega_a y\,,$$

with x and y arbitrary spinors for each mode $(\boldsymbol{p}, r, a)$. The normalization will be made later.

Provisionally we shall define the neutron-antineutron field ψ_n and the proton-antiproton field ψ_p by

$$(2.18)\qquad \left(\frac{\beta_7+1}{2}\right)\psi_p\equiv\Omega_+\psi_p=\psi_p\,,\qquad \left(\frac{-\beta_7+1}{2}\right)\psi_n\equiv\Omega_-\psi_n=\psi_n\,.$$

Thus for the quantum fields with mass $\varkappa$ (units $\hbar=c=1$) we have

$$(2.19a)\quad \begin{cases} \psi_p = U(x)N(\varkappa)f_+(\lambda)\omega_p(x)\,,\\ \omega_p(x)=\left(\dfrac{1}{L}\right)^3\sum\limits_{p,r}\left(\dfrac{\varkappa}{p^4}\right)^{\frac{1}{2}}\left[u_r^+(p)\exp[ipx]a_r^+(p)+v_r^+(p)\exp[-ipx]b_r^+(p)^*\right] \end{cases}$$

and

$$(2.19b)\quad \begin{cases} \psi_n = U(x)N(\varkappa)f_-(\lambda)\omega_n(x)\,,\\ \omega_n(x)=\left(\dfrac{1}{L}\right)^3\sum\limits_{p,r}\left(\dfrac{\varkappa}{p^4}\right)^{\frac{1}{2}}\left[u_r^-(p)\exp[ipx]a_r^-(p)+v_r^-(p)\exp[-ipx]b_r^-(p)^*\right], \end{cases}$$

where $N(\varkappa)$ is a normalization to be determined later.

Thus for a given R, ψ_p and ψ_n define two *equivalent* 4-dimensional irreducible representations of M_{prop}, the proper subgroup of M [7]. They are irreducible since we know that there are no irreducible spinor representations of M_{prop} of dimension less than 4, and equivalent since they possess the same set of invariant eigenvalues $Q(R^2)$, R, $W(R^2)$. The irreducible representations of M_{prop} occur in pairs $(Q, \pm R, W)$ for $R \neq 0$, as is the case here, because R_{0p} is pseudoscalar whereas Q_{0p} and W_{0p} are scalar. Thus for $-R$ we get another pair of equivalent 4-dimensional irreducible representations of M_{prop}, and neither of these is equivalent under M_{prop} to either of the pair (2.19), since the eigenvalue sets of the invariants are different. Now we can build an irreducible (8-dimensional) representation of M by forming the direct sum of $\psi_n(-|R|)$ and $\psi_p(|R|)$. This is indeed a representation of M, for under any improper transformation of M, $R \to -R$ and $\beta_7' \to -\beta_7'$ so that, bearing (2.18) in mind, we see that $\{n, \bar{n}\}$ and $\{p, \bar{p}\}$ fields are simply interchanged. It is irreducible since 8-dimensional. This field, which furnishes an irreducible representation of M, we call the nucleon-antinucleon field for the mass $\varkappa$, ψ. In virtue of

$$\{f_{\pm}(\lambda)\}_R = \{f_{\mp}(\lambda)\}_{-R} \tag{2.20}$$

it becomes

$$\begin{cases} \psi = U(x)g(\lambda)\omega(x)\,, \\ \omega(x) = \left(\dfrac{1}{L}\right)^{\frac{3}{2}} \sum_{p,r,a} \left(\dfrac{\varkappa}{p^4}\right)^{\frac{1}{4}} [u_r^a(p) \exp[ipx] a_r^a(p) + v_r^a(p) \exp[-ipx] b_r^a(p)^*], \end{cases} \tag{2.21}$$

with

$$g(\lambda) \equiv N(\varkappa)\{f_+(\lambda)\}_{R>0} = N(\varkappa)\lambda^2\{J_{\nu_-}(\varkappa\lambda)Q_1 + (-1)^{\nu_+}\lambda J_{\nu_+}(\varkappa\lambda)Q_2\}. \tag{2.22}$$

For notational simplicity the index p in $u_r^a(p)$ and $a_r^a(p)$ etc., may be understood to include the momentum $\boldsymbol{p}$ and the mass $\varkappa$.

The complete nucleon-antinucleon field Ψ should be a sum of these partial fields over the allowed mass spectrum $\varkappa_i$ $(i = 1, 2, \ldots)$ (see Section 3 for a discussion and justification of this point). For reasons which will emerge later we choose the irreducible representation with $|R| = \frac{9}{8}$ to represent the nucleon:

$$\nu_+ = 1\,, \qquad \nu_- = 0 \qquad \left(R = \tfrac{9}{8}\right) \tag{2.23}$$

[7] This discussion always referes to the c-number fields, *i.e.*, the amplitudes of the quantum field.

and choose the mass spectrum and normalization $N(\varkappa)$ to be as follows

$$\text{(2.24)} \qquad \begin{cases} J_1(\varkappa_i l) = 0\,, \qquad \varkappa_i l > 0 & (i = 1, 2, \ldots), \\ N(\varkappa_i) = \left[\dfrac{2}{l^2 J_0^2(\varkappa_i l)}\right]^{\frac{1}{2}}, \end{cases}$$

where l defines the range of λ: $0 < \lambda \leqslant l$. Thus the complete nucleon field is

$$\text{(2.25)} \qquad \Psi = \sum_{\varkappa_i} \psi_{\varkappa_i}\,,$$

with

$$\text{(2.26)} \qquad g_i(\lambda) = \left[\frac{2}{l^2 J_0^2(\varkappa_i l)}\right]^{\frac{1}{2}} \lambda^2 \{J_0(\varkappa_i \lambda) Q - \lambda J_1(\varkappa_i \lambda) Q_2\}\,.$$

Adjoint. – With A the negative hermitizing spin operator

$$\text{(2.26)} \qquad A\beta_\mu A^{-1} = -\beta_\mu^\dagger\,, \qquad A^\dagger = A\,,$$

we form the adjoint [8] $\bar{\psi} = \psi^* A$. We shall need the behavior of the various spin-operators introduced up to now under hermitian conjugation:

$$\text{(2.27)} \qquad \begin{cases} \beta_i^\dagger A = A\beta_7\,, \qquad M^\dagger A = -AM\,, \\ \Lambda_\pm(p)^\dagger A = A\,\Lambda_\pm(p)\,, \qquad \Sigma_r^\dagger A = A\Sigma_r\,, \qquad \Omega_\pm^\dagger A = A\Omega_\pm\,, \\ Q_1^\dagger A = AQ_2\,, \qquad Q_2^\dagger A = AQ_1\,. \end{cases}$$

Then it follows

$$\text{(2.28)} \qquad \bar{\psi} = \bar{\omega}(x)\, h(\lambda)(1 + \tfrac{1}{2}\beta x \beta_5) = \bar{\omega}(x)\, h(\lambda) U^{-1}(x)\,,$$

with

$$\text{(2.29)} \qquad h(\lambda) \equiv N(\varkappa)\lambda^2 \{J_0(\varkappa\lambda) Q_2 - \lambda J_1(\varkappa\lambda) Q_1\}$$

and

$$\text{(2.30)} \qquad \bar{\omega}(x) = \left(\frac{1}{L}\right)^{\frac{3}{2}} \sum_{p\, r.a} \left(\frac{\varkappa}{p^3}\right)^{\frac{1}{2}} [\bar{u}_r^a(p) \exp[-ipx] a_r^a(p)^* + \bar{v}_r^a(p) \exp[ipx] b_r^a(p)]\,,$$

with $\bar{u} = u^* A$, $\bar{v} = v^* A$.

[8] The star * means hermitian conjugate for the operators of the quantized field and complex conjugate for c-numbers, spinor amplitudes, and so on.

Normalization. – The basic spinors $u_r^a(p)$ and $r_r^a(p)$ in (2.21) are orthogonal and understood to be normalized as follows

$$\overline{u}_r^{\pm} u_r^{\pm} = \pm 1, \qquad \overline{v}_r^{\pm} v_r^{\pm} = \mp 1 . \tag{2.31}$$

These are for a given $\boldsymbol{p}$ and $\varkappa$. This normalization may be attained as follows. For a given mode $\boldsymbol{p}$ and $\varkappa$, with $\boldsymbol{p}$ aligned along Oz for convenience, start by defining

$$u_{\ddagger}^{\pm} \equiv \Lambda_- \Sigma_- \Omega_+ x \tag{2.32}$$

and choose the spinor x and the sign of Λ such that $u_-^- u_-^+ = +1$. Then define successively

$$\begin{cases} u_-^+ \equiv \beta_1 P u_+^+ , & P \equiv \beta_0 - \tfrac{1}{2}\beta_5 , \\ u_r^- \equiv P u_r^+ , & (r = \pm), \\ v_r^a \equiv \gamma_5 u_r^a , & \gamma_5 \equiv i\beta_1\beta_2\beta_3\beta_4 , \qquad (r = \pm\ ,\ a = \pm). \end{cases} \tag{2.33}$$

That these basic spinors have indeed the proper quantum numbers indicated and realize the normalization (2.31) can be proved in the usual way [9] by making use of the commutation properties of these spin operators with the basic projections and of their behavior under hermitian conjugation. The pattern of sign in (2.31) is unique – – – – – – – – for instance, having chosen $\overline{u}^+ u^+ = +1$, it is impossible to attain also $\overline{u}^- u^- = +1$.

3. – Mass and the physical meaning of λ.

The preceding solutions for the wave functions show that the new co-ordinate λ and the mass are conjugate in the usual sense [10].

λ is a freely assignable length, that is, *a coordinate with the (one-dimensional) domain of all lengths lying in the interval* $(0, l)$. (l might possibly be ∞, whence λ would have the domain of all lengths.) It is in no sense any particular, fixed invariant length in conformal physical theory. Now each mode in the plane wave expansion (2.21) possesses a definite, sharp value of the mass, $\varkappa$. Hence the distribution in λ, (2.26), is as broad as it in principle can be. We mean this to be precisely analogous to the p, x conjugacy: the 4-momentum of each mode in (2.21) is perfectly sharp, hence the position of the quantum in space-

[9] See J. JAUCH and F. ROHRLICH: *Theory of Photons and Electrons* (Cambridge, 1955), Appendix A2-3.

[10] This idea is already implicit in the theory of reference [13] and has also been expressed by MURAI in a letter to the author (June 1959).

time is as indeterminate as it in principle can be. However, due to the qualitative difference between momentum and mass, the broadest possible distributions in position $|\exp[\pm ipx]|^2 = 1$ are completely uniform, whereas the same thing in λ, $J^2_{1.0}(\varkappa\lambda)$, show gentle maxima in the region $\lambda \simeq \varkappa^{-1}$ and go to zero slowly as $\varkappa\lambda \to \infty$. We can understand this simply by observing that ordinary space is (for a free particle) completely homogeneous; not so length space, which shows definite inhomogeneities, definite landmarks, due to the existence of important invariant lengths in nature, such as the Compton wave lengths of the various elementary particles, etc.

It might seem tempting to restrict our quantum field to a single mass mode $\varkappa_0$, say. However, this does not seem natural to us for several reasons.

1) A realistic use of free particle theory makes use of wave packets, more or less localized in space, hence with an inevitable spread in momentum. Now it might be physically possible and desirable to form a wave packet also in length space, which would mean some more or less sharp localization of the measurable length λ associated with a nucleon. This would require a spread in $\varkappa$, that is, a particle mode with no precise mass. Similarly we might expect that interaction with other fields, which destroys the constancy of the 4-momentum of a particle, would if λ-dependent likewise destroy the perfect constancy of its mass.

2) The connection of $\varkappa$ (defined for a *real free* nucleon of 4-momentum p_m by $p^2 + \varkappa^2 = 0$) with its *observed* mass M_N seems not entirely clear. For experiments which result in a measurement of elementary particle mass usually involve interactions with external fields (localization of the particle in space, *i.e.*, the formation of a tight wave packet, in particular requires such strong interactions). For example, in the mass-spectrographic method of determining M_N, there is the comparatively weak interaction with an applied E.M. field and strong interaction with emulsion molecules. Thus the act of measuring the mass would introduce other values $\varkappa \neq \varkappa_0$ by « scattering » in length space if either the external interacting fields were λ-dependent or if the act of measurement implied forming a wave packet in length space. What seems to be needed is a careful Bohr-Rosenfeld type examination of the physical process of measuring elementary particle mass, with the aim of discovering whether there is any measurable length attributable to the particle which becomes more imprecise, the more precisely the observed mass is known, and vice-versa. The big mathematical problem is then to explain how, in virtue of external interactions, only the real nucleon modes with $\varkappa \simeq M_N$ result occupied.

3) The usefulness of a bare particle mass spectrum in taking into account the interaction of the nucleon with the virtual quanta of the various fields to which it is coupled seems fairly clear from the example of the theory of

regulators. This « taking into account » we may call « renormalization » for this theory (although it may differ qualitatively from the renormalization of the usual theories). Whether the so renormalized theory should still admit a (real) mass spectrum, as assumed in point 2) above, or contain only the single, observed mass $M_{\mathcal{N}}$ is an open question.

Summing up, we can say the following. The fundamental group M allows us to treat particles whose mass can be unsharp, and provides a definite formalism for treating this problem in which, analogous to p, x conjugacy, there is a $\varkappa$, λ, or mass, fundamental length, conjugacy. The *complete* equivalence of momentum and mass, which would be physical nonsense contradicting experience, is not asserted — the theory shows a qualitative difference between the two which comes down to the fact that whereas ordinary space is to be idealized as homogeneous, length space is not.

4. – Total energy, momentum, angular momentum and charge.

Consider the action integral

$$I = \int \mathrm{d}\Omega\, \mathscr{L}\,, \qquad \mathscr{L} \equiv \frac{4}{9}\,\overline{\Psi}(R_{0p} - \beta_7 R)\Psi\,, \qquad (R > 0). \tag{4.1}$$

We integrate over a solid angle region in the 6-dimensional space, the element of solid angle being

$$\begin{cases} \mathrm{d}\Omega \equiv \sqrt{\mathscr{S}}\, \varepsilon_{\mu\nu\lambda\xi\varrho\tau}\, X^{\mu}\, \mathrm{d}X_1^{\nu} \ldots \mathrm{d}X_5^{\tau}\,, \\ \mathscr{S} = \mathrm{Det}\, \mathscr{S}_{\mu\nu}\,, \qquad \mathscr{S}_{\mu\nu} = -\dfrac{g_{\mu\nu}}{g_{\lambda\xi} X^{\lambda} X^{\xi}}\,. \end{cases} \tag{4.2}$$

(This corresponds exactly to integrating over some ordinary region of the inhomogeneous co-ordinates x^{m}, λ and in fact $\mathrm{d}\Omega = \lambda^{-5}\, \mathrm{d}^4x\, \mathrm{d}\lambda$.) $\mathrm{d}\Omega$ is solid angle *with* sense, thus a pseudoscalar. Since $\mathscr{L}$ is also pseudo scalar, I is an invariant. Variation of $\overline{\Psi}$ will give the nucleon wave equation

$$R_{0p}\Psi = \beta_7 R\Psi \qquad (R > 0) \tag{4.3}$$

which is verified for both the neutron ($\Psi_{\mathrm{n}}(-|R|)$, $\beta_7' = -1$) and proton ($\Psi_{\mathrm{p}}(|R|)$, $\beta_7' = +1$) components of Ψ. $\mathscr{L}$ being hermitian, the adjoint equation will be consistent with (4.3).

We want to formulate the constants of the motion of the total field as the generators of the unitary transformations induced on the state Hilbert space by the various symmetry transformations of the theory. If $\delta\Psi = \Psi'(X) - \Psi(X)$ is the formal variation in Ψ due to one of these symmetry transformations, it is shown in Appendix I via an angular Gauss's Theorem

that if the field equations are satisfied, then

$$(4.4)\qquad \begin{cases} \delta\int \mathrm{d}\Omega\, \mathscr{L} = F(f) - F(f_0)\,, \\ F(f) = \int\limits_f \mathrm{d}f_\xi\, X_\lambda \overline{\Psi}\beta_7\beta^{[\lambda}\beta^{\xi]}\,\delta\Psi\,, \end{cases}$$

where f and f_0 are certain bounding surfaces defined there, which can be taken to be two surfaces $x^4 = \text{const.}$ in particular. By an action principle we postulate that the corresponding infinitesimal unitary transformation $U(f)$ on the surface f is just exactly

$$(4.5)\qquad U(f) = \mathbf{1} + iF(f)\,.$$

The normalization of $\mathscr{L}$ has been chosen accordingly. If the variation is due to the 6-rotation

$$(4.6)\qquad X'^\mu = X^\mu + \varepsilon^\mu{}_\nu X^\nu\,, \qquad \varepsilon_{\mu\nu}(\equiv g_{\mu\lambda}\varepsilon^\lambda{}_\nu) = -\varepsilon_{\nu\mu} \quad \text{real, infinitesimal,}$$

the variation is $\delta\Psi = (i/2)\, M_{\mu\nu}\varepsilon^{\mu\nu}\Psi$ with $M_{\mu\nu}$ the total generator (2.6) *cum* (2.5). Writing $F(f) = (1/2)\mathscr{M}_{\mu\nu}\varepsilon^{\mu\nu}$ in this case, we get

$$(4.7)\qquad \mathscr{M}_{\mu\nu} = \int\limits_f \mathrm{d}f_\xi\, X_\lambda \overline{\Psi}\beta_7\beta^{[\lambda}\beta^{\xi]}\, iM_{\mu\nu}\Psi\,.$$

These fifteen $\mathscr{M}_{\mu\nu}$ must then represent various dynamical constants of the motion of the total nucleon field.

To identify these we must simply see to what 6-rotations the familiar space-time transformations correspond. From the connection (2.7) of x and λ with the X^μ it follows that the infinitesimal space-time translations, Lorentz rotations, and dilatations, respectively

$$(4.8)\qquad \begin{cases} x'^m = x^m + a^m\,, & \lambda' = \lambda\,, \\ x'^m = x^m + b^m{}_n x^n\,, & \lambda' = \lambda\,; \qquad b_{mn} = -b_{nm}\,, \\ x'^m = x^m + cx^m\,, & \lambda' = \lambda + c\lambda \end{cases}$$

are represented by the infinitesimal 6-rotations $\varepsilon^\mu{}_\nu$ with

$$(4.9)\qquad \begin{cases} \text{translation:} & \varepsilon^m{}_0 = a^m\,, & \text{other } \varepsilon^\mu{}_\nu \text{ zero}\,. \\ \text{rotation:} & \varepsilon^m{}_n = b^m{}_n\,, & \text{»} \quad \text{»} \quad \text{»}\,. \\ \text{dilatation:} & \varepsilon^0{}_0 = -c\,, & \text{»} \quad \text{»} \quad \text{»}\,. \end{cases}$$

Here « other ε''_{ν} zero » means that the other ε''_{ν} independent of the specified ones vanish. For instance, for the translation subgroup $\varepsilon^5{}_n = g_{nm}\varepsilon^m{}_0 = a_n \neq 0$, and so on. (4.9) shows clearly the convenience of the non principal axes fundamental form (2.3). Hence $\mathscr{M}_{5m}$ and $\mathscr{M}_{mn}$ generate translations and rotations respectively and should be equal to the total 4-momentum P_m and total 4-angular momentum J_{mn} of the nucleon field:

$$P_m = \mathscr{M}_{5m}\,, \qquad J_{mn} = \mathscr{M}_{mn}\,. \tag{4.10}$$

We start by proving the first equality.

In the integrand of $\mathscr{M}_{5m}$ we encounter terms like

$$i(\overset{0}{M}_{5m} + S_{5m})\{U(x)\,g(\lambda)\,\omega(x)\}.$$

The action of the orbital parts $\overset{0}{M}_{\mu\nu}$ on functions of x and λ is given by

$$\begin{cases} i\overset{0}{M}_{mn}x^p = x_m\delta^p_n - x_n\delta^p_m\,, & i\overset{0}{M}_{mn}\lambda = 0\,, \\ i\overset{0}{M}_{5m}x^n = -\delta^n_m\,, & i\overset{0}{M}_{5m}\lambda = 0\,, \\ i\overset{0}{M}_{0m}x^n = x_m x^n - \delta^n_m\tfrac{1}{2}(x^2+\lambda^2)\,, & i\overset{0}{M}_{0m}\lambda = x_m\lambda\,, \\ i\overset{0}{M}_{50}\,x^m = x^m\,, & i\overset{0}{M}_{50}\lambda = \lambda\,. \end{cases} \tag{4.11}$$

From (4.11) and the commutation properties of the β_μ the important commutation relation

$$i(\overset{0}{M}_{5m} + S_{5m})U(x) \equiv \left(-\frac{\partial}{\partial x^m} + \frac{1}{2}\beta_5\beta_m\right)U(x) = U(x)\frac{\partial}{\partial x^m}\,, \tag{4.12}$$

can be proved. Thus the spin part cancels out entirely and since also

$$\left[\frac{\partial}{\partial x^m}\,,\ g(\lambda)\right] = 0\,, \tag{4.13}$$

it turns out that M_{5m} is the expected differential operator $(-1/i)\,\partial/\partial x^m$ on the x-dependent part of the « renormalized » field $\omega(x)$. This obviates the difficulty of the apparently anomalous « translation spin » [11]. Thus we get

$$\mathscr{M}_{5m} = \sum_{i,j}\int \mathrm{d}f_\xi\,\overline{\omega}_i(x)\,h_i(\lambda)\,U^{-1}(x)\,X_\lambda\,\beta^{[\lambda}\beta^{\xi]}U(x)g_j(\lambda)\beta_7\left(-\frac{\partial}{\partial x^m}\right)\omega_j(x)\,. \tag{4.14}$$

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[11] R. INGRAHAM: *Phys. Rev.*, **106**, 595 (1957).

From now on for simplicity we treat only the component ψ belonging to the mass $\varkappa$ and at the end add up these separate contributions. The different modes $\varkappa_i$ never interfere because of the orthogonality of $J_0(\varkappa_i\lambda)$ and $J_0(\varkappa_j\lambda)$ and of $J_1(\varkappa_i\lambda)$ and $J_1(\varkappa_j\lambda)$ when $\varkappa_i l \neq \varkappa_j l$ are positive zeroes of $J_1(z)$ as required in (2.24). Now for the surface f: $x^4 = \text{const.}$ it is shown in Appendix II that

$$\mathrm{d}f_\xi U^{-1}(x) X_\lambda \beta^{[\lambda} \beta^{\xi]} U(x) = -\,\mathrm{d}^3x\,\mathrm{d}\lambda\,\lambda^{-5}\beta^4\left(\beta_0 + \frac{\lambda^2}{2}\beta_5\right). \tag{4.15}$$

When this is substituted into (4.14), we get the following expression

$$h(\lambda)\beta^4\left(\beta_0 + \frac{\lambda^2}{2}\beta_5\right) g(\lambda) = \beta^4 h(\lambda)\left(\beta_0 + \frac{\lambda^2}{2}\beta_5\right) g(\lambda) = \tag{4.16}$$

$$= \lambda^6\beta^4 N^2\left\{J_1^2(\varkappa\lambda)\beta_0 + J_0^2(\varkappa\lambda)\frac{1}{2}\beta_5\right\}.$$

The last equality follows easily from the properties of the projections Q_1 and Q_2

$$Q_2\beta_5 Q_1 = \beta_5\,, \qquad Q_1\beta_0 Q_2 = \beta_0\,, \tag{4.17}$$

while all other products $Q_i\beta_0 Q_j$, $Q_i\beta_5 Q_j$ $(i, j = 1, 2)$ vanish. The λ integration can now be done. Because

$$\int_0^l \mathrm{d}\lambda\,\lambda\,J_0(\varkappa_i\lambda)\,J_0(\varkappa_i\lambda) = \int_0^l \mathrm{d}\lambda\,\lambda\,J_1(\varkappa_i\lambda)\,J_1(\varkappa_j\lambda) = \frac{l^2}{2}J_0^2(\varkappa_i l)\,\delta_{ij} = [N(\varkappa_i)]^{-2}\delta_{ij}\,, \tag{4.18}$$

$\int_0^l \mathrm{d}\lambda\,\lambda^{-5}$ applied to the right member of (4.16) becomes simply $\beta^4(M/\varkappa)$ where M is the mass operator, and we get

$$\mathscr{M}_{5m} = \int \mathrm{d}^3x\,\bar{\omega}(x)\beta^4\frac{M}{\varkappa}\beta_7\frac{\partial}{\partial x^m}\omega(x)\,. \tag{4.19}$$

The plane wave expansions are now substituted. After x integration over the periodicity cube L^3 we are left with

$$\begin{aligned}\mathscr{M}_{5m} = \sum\left(\frac{\varkappa}{p^4}\right) ip_m a\{&\bar{u}(p)\Gamma^4 u(p)\,a(p)^* a(p) - \\ &- \bar{v}(p)\Gamma^4 v(p)\,b(p)b(p)^* + \bar{v}(-p)\Gamma^4 u(p)\,b(-p)\,a(p)\exp\left[2ip_4x^4\right] - \\ &- \bar{u}(-p)\Gamma^4 v(p)\,a(-p)^* b(p)^* \exp\left[-2ip_4x^4\right]\},\end{aligned} \tag{4.20}$$

where we have set $\Gamma^4 = \beta^4(M/\varkappa)$ and suppressed the spin and handedness indices for notational simplicity. a = eigenvalue of β_7 on the wave functions u and v. The sum is over $\boldsymbol{p}$, r_1, r_2, a_1 and $a_2 = a$. Γ^4 has commutation properties with the projections $\Lambda_\pm$ analogous to those of γ^4 with the corresponding projections in electron theory. In fact it is easy to prove

$$\Lambda_\pm(p)\Gamma^4 = \Gamma^4\Lambda_\mp(p) \pm i\frac{p^4}{\varkappa}, \tag{4.21a}$$

$$\Lambda_\pm(p)\Gamma^4 = \Gamma^4\Lambda_\pm(-p). \tag{4.21b}$$

The u, v cross terms in (4.20) go out in the usual way; for example, using (4.21b) we can write (for certain spinors x and y)

$$\bar{v}(-p)\Gamma^4 u(p) = y\Lambda_+(-p)\Gamma^4\Lambda_-(p)x = y\Gamma^4\Lambda_+(p)\Lambda_-(p)x = 0.$$

For the remaining spin-scalars we can prove

$$a_2\bar{u}^{a_1}_{r_1}(p)\Gamma^4 u^{a_2}_{r_2}(p) = a_2\bar{v}^{a_1}_{r_1}(p)\Gamma^4 v^{a_2}_{r_2}(p) = -i\left(\frac{p^4}{\varkappa}\right)\delta_{r_1r_2}\delta_{a_1a_2}, \tag{4.22}$$

using (4.21a). For instance $\bar{u}\Gamma^4 u$, $\bar{v}\Gamma^4 v$ involve the products $\Lambda_\mp(p)\Gamma^4\Lambda_\mp(p)$, which can be manipulated as follows

$$\Lambda_\mp(p)\Gamma^4\Lambda_\mp(p) = \left(\Gamma^4\Lambda_\pm(p) \mp i\frac{p^4}{\varkappa}\right)\Lambda_\mp(p) = \mp i\left(\frac{p^4}{\varkappa}\right)\Lambda_\mp(p) = \mp i\left(\frac{p^4}{\varkappa}\right)\Lambda_\mp(p)\Lambda_\mp(p),$$

so that effectively the scalars $\mp i(p^4/\varkappa)$ take the place of Γ^4 in the respective spin-scalars. Together with the normalization (2.31) this proves (4.22). Putting these results into (4.20), we come out with

$$\mathscr{M}_{5m} = \sum_{p,r,a} p_m\left(n^a_r(p) + \bar{n}^a_r(p)\right) = P_m, \qquad \textit{q.e.d.}, \tag{4.23}$$

where

$$n^a_r(p) \equiv a^a_r(p)^* a^a_r(p), \qquad \bar{n}^a_r(p) \equiv b^a_r(p)^* b^a_r(p) \tag{4.24}$$

are the number operators for particle and antiparticle modes and the ordering convention on the fermion operators $:bb^*: = -b^*b$ has eliminated the minus in the antiparticle term.

To show that we also get the correct ordinary angular momentum we must verify that $M_{ij} = \overset{0}{M}_{ij} + S_{ij}$ $(i, j = 1, 2, 3)$ has the correct commutation properties with the spin-operators $U(x)$ and $g(\lambda)$. These can be shown to be

$$[M_{ij}, U(x)] = [M_{ij}, g(\lambda)] = 0. \tag{4.25}$$

Thus we can take over the analysis for the linear momentum by simply replacing $-\partial/\partial x^m$ by $i\overset{0}{M}_{ij}+iS_{ij}$ in the formula (4.14) and the following. This obviously leads to the correct result, for instance, we get the longitudinal spin angular momentum of a single mode, $S_{\parallel}(p)$ say, by the eigenvalue replacements $p_m \to r/2$ on both modes u_r and v_r (remembering (2.15)). Hence from (4.23) we can write

$$S_{\parallel}(p) = \sum_{r,a} \frac{r}{2} \left(n^a_r(p) + \overline{n}^a_r(p)\right), \tag{4.26}$$

which is just exactly the longitudinal component of the spin angular momentum of the mode $\boldsymbol{p}$ (particle or antiparticle) with $r = \pm 1$ signifying the the value $\pm \frac{1}{2}$ of this component in the direction of motion, *q.e.d.*. The constants of motion of the complete field are now obtained by summing the partial constants of the motion over the mass spectrum.

The question of the physical meaning of the other dynamical constants, of the motion is an interesting one, still open at the moment. All of these $\mathscr{M}_{0m}$ and $\mathscr{M}_{50}$ share with the orbital angular momentum L_{ij} $(i, j = 1, 2, 3)$ the property of not being sharp in a basis of plane waves. Just as $\mathscr{M}_{4i}$ is no actual angular momentum, but rather its time constancy enunciates a center of mass motion theorem, we expect that the four time constant $\mathscr{M}_{0m}$ will give some sort of motion theorem having to do with uniform acceleration. The most interesting new constant of the motion is perhaps the Lorentz invariant $\mathscr{M}_{50}$, the generator of the unitary subgroup induced by the dilatations. *The dilatation subgroup*

$$x'^m = \mu x^m, \qquad \lambda' = \mu\lambda\,; \qquad \mu = \text{const} > 0 \tag{4.27}$$

of the group M means nothing more or less than changes of the unit of length (say from cm to m). *Any* invariantly defined length, say the observed nucleon Compton wave length $\lambda\!\!\!^{-}_{\mathcal{N}}$, suffers the same transformation: $\lambda\!\!\!^{-\prime}_{\mathcal{N}} = \mu\lambda\!\!\!^{-}_{\mathcal{N}}$ on the dilatation (4.27), since it is just a particular fixed point $\lambda_0 = \lambda\!\!\!^{-}_{\mathcal{N}}$ in length space. By (4.27) *all* points λ in length space suffer this transformation, thus in particular any one point λ_0 does so [12]. We must emphasize that the notion that (4.27) somehow means « actual changes of length » (whatever that means) and therefore is in conflict with microphysics since there exist fixed universal lengths in nature, etc., is not only superficial but wrong. Now unlike M_{5m},

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[12] Consider the analogous statement for ordinary space. If $\boldsymbol{r}_0$ is any invariantly defined point in ordinary space, let us say the tip of the Eiffel Tower, then under the 3-translation $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}$ we have also $\boldsymbol{r}'_0 = \boldsymbol{r}_0 + \boldsymbol{a}$. For the first equation says that *all* points of ordinary space suffer this transformation, hence any particular point $\boldsymbol{r}_0$ does so.

M_{50} acts without change on the inner fields $g(\lambda)\omega(x)$ since

$$[M_{50}, U(x)] = 0 . \tag{4.28}$$

The spin part acting on $g(\lambda)$ changes it into a similar λ-dependent operator since

$$iS_{50}Q_1 = \tfrac{1}{2}Q_1, \qquad iS_{50}Q_2 = -\tfrac{1}{2}Q_2 . \tag{4.29}$$

The orbital part $i\overset{0}{M}_{50} = x\,\partial/\partial x + \lambda\,\partial/\partial\lambda$ would be sharp only in a basis of elementary wave functions homogeneous of some degree in x and λ. Our plane wave functions $\sim \lambda^q J_{1.0}(\varkappa\lambda)\exp[\pm ipx]$ do not satisfy this condition. Therefore the physical meaning of $\mathscr{M}_{50}$ could probably be found only if one could extract a set of homogeneous elementary solutions of Murai's wave equation. It seems certain that the linear momentum would be not even nearly sharp in these modes.

Charge and Baryon number. – Our Lagrangian (4.1) admits the symmetry group

$$\Psi' = \exp[i\alpha\Omega_+]\Psi , \qquad \overline{\Psi}' = \overline{\Psi}\exp[-i\alpha\Omega_+] \qquad (\alpha \text{ real}) \tag{4.30}$$

or $\delta\Psi = i\varepsilon\Omega_+\Psi$ with ε real and infinitesimal. The corresponding generator is by (4.4)

$$G(f) = \int \mathrm{d}f_{\xi}\, X_{\lambda}\overline{\Psi}\beta_7\beta^{[\lambda}\beta^{\xi]}\, i\Omega_+\Psi . \tag{4.31}$$

The commutation of Ω_+ with $U(x)$ and the $g(\lambda)$ is trivial, so Ω_+ acts directly on the space-time parts $\omega(x)$ of the nucleon field, where it picks out the protonic component ($a = +1$) only. Thus we can take over the analysis given in detail for the linear momentum by making the eigenvalue substitutions

$$p_m \to \begin{cases} +1 \\ -1 \end{cases} \quad \text{for the} \quad \begin{cases} \text{antiparticle} \\ \text{particle} \end{cases} \quad \text{mode,} \quad a = +1 .$$

$$p_m \to 0 \qquad \text{for the modes} \qquad a = -1 .$$

This gives

$$G = \sum_{p,r}\{n_r^+(p) - \overline{n}_r^+(p)\} = Q , \qquad \textit{q.e.d.} . \tag{4.32}$$

Q is just the nucleon charge operator in units of e (> 0) the elementary charge with the identification (2.18) of the $\{p, \overline{p}\}$ and $\{n, \overline{n}\}$ fields. The symmetry

group of constant phase transformations

$$(4.33)\qquad \begin{cases} \Psi' = \exp[i\alpha]\Psi\,, \qquad \overline{\Psi}' = \exp[-i\alpha]\overline{\Psi} & (\alpha \text{ real}) \\ \delta\Psi = i\varepsilon\Psi & (\varepsilon \text{ real, infinitesimal}) \end{cases}$$

with generator

$$(4.34)\qquad H(f) = \int \mathrm{d}f_\xi X_\lambda \overline{\Psi}\beta_7\beta^{[\lambda}\beta^{\xi]}i\Psi\,,$$

will likewise yield the baryon number

$$(4.35)\qquad H = \sum_{p,r}\{n_r^+(p) - \overline{n}_r^+(p) + n_r^-(p) - \overline{n}_r^-(p)\} = N\,.$$

The same idea of defining the neutron and proton states by the eigenvalues ± 1 of γ_5 occurs in the Gürsey baryon theory. We want to emphasize that the extra symmetry (4.30) (*i.e.*, besides the co-ordinate symmetry group and the phase transformations (4.33) always present in quantum mechanics) is an automatic feature of the irreducible representations of the group M, while this (*i.e.*, with $\beta_7 \to \gamma_5$) is not so for its subgroup L_{inh}. One has to go to *reducible* representations of L_{inh} to secure this symmetry, except for the case of zero mass.

5. – Various concluding remarks.

If the above identification of the nucleon charge is correct, then the electromagnetic potentials, properly phrased in the framework of the group M, would have to be coupled to the nucleon field with the factor Ω_+ to guarantee the inertness of n and $\overline{n}$ under electrostatic forces. The magnetic moment of the neutron, as calculated in the c-number theory, would then also be zero. The M-covariant theory of the photon field could well be that given in reference [13]. Its interpretation as a « family of bosons » with spins 1 and 0 and a certain mass spectrum should be rejected in favor of the present interpretation of a single field, the photon field, with a bare particle mass spectrum.

Granting the physical correctness of this identification of Q for the sake of argument, there is still the question of the proper correlation of β_7' with the neutron and proton. The opposite correlation to (2.18) is of course possible. The resulting irreducible 8-dimensional nucleon field would presumably

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[13] R. Ingraham and J. Ford: *Phys. Rev.*, **101**, 1411 (1956).

not be essentially different from the one chosen because of the unicity of the 8-dimensional irreducible spinor representation of M up to equivalence.

The length l defining the range of λ, $0 < \lambda \leqslant l$, is a parameter about whose size we have at the moment no clue. If however, we keep it finite, hence treat a discrete mass spectrum, the pair of representations $R = \pm 27/8$ seems to give a qualitatively different theory from the one with $R = \pm 9/8$. For since $\nu_+ = 2$, $\nu_- = -1$ for $R = 27/8$, we are dealing with the Bessel functions J_1 and J_2. In this case it is impossible to define a spectrum $\varkappa_i l$ such that different modes $J_\nu(\varkappa_i\lambda)$, $J_\nu(\varkappa_j\lambda)$, $\varkappa_i \neq \varkappa_j$, are orthogonal for both $\nu = 2$ and 1 and such that also

$$\int_0^l \mathrm{d}\lambda\,\lambda J_1^2(\varkappa_i\lambda) = \int_0^l \mathrm{d}\lambda\,\lambda J_2^2(\varkappa_i\lambda) \qquad (i = 1, 2, \ldots).$$

Thus we do *not* come out with a spin-operator $\propto M$ after doing the λ-integration in (4.16). The resulting new Γ^4 no longer has the simple properties (4.22), and as a result there is interference between the plane wave modes. This would seem to reject the $|R| = 27/8$ irreducible representation of M as unphysical, leaving us with the unique representation $|R| = 9/8$.

Interactions. – Certain qualitative features of the interaction of this field with other M-covariant fields can already be seen. Consider the M-covariant photon field cited above ([13]). The components there depend on λ through factors $J_0(\varkappa\lambda)$ and $J_1(\varkappa\lambda)$ and the mass spectrum may be taken to be the same as in this theory. Consider interaction with the part $\propto J_0(\varkappa\lambda)$ to fix ideas. At the vertex of the elementary graph we should encounter the integrals

$$\text{(5.1)} \qquad \begin{cases} \displaystyle\int_0^\infty \mathrm{d}\lambda\,\lambda J_0(\varkappa'\lambda)J_0(\varkappa''\lambda)J_0(\varkappa\lambda) \equiv g_0(\varkappa', \varkappa'', \varkappa)\,, \\ \displaystyle\int_0^\infty \mathrm{d}\lambda\,\lambda J_1(\varkappa'\lambda)J_1(\varkappa''\lambda)J_0(\varkappa\lambda) \equiv g_1(\varkappa', \varkappa'', \varkappa)\,. \end{cases}$$

(We have let $l \to \infty$, getting a continuous mass spectrum.) Then one can show

$$\text{(5.2)} \qquad g_\nu(\varkappa', \varkappa'', \varkappa) = 0 \quad \text{unless} \quad |\varkappa' - \varkappa''| < \varkappa < \varkappa' + \varkappa'' \qquad (\nu = 0, 1)$$

so that interaction with photon mass modes $\varkappa$ is restricted to a certain finite range depending on the pair of fermion mass modes $\varkappa'$, $\varkappa''$ in question. (In virtue of (5.2) the g_ν might be called « continuous Clebsch-Gordan coefficients ».) In-

side the permitted range we have, for example

$$g_0(\varkappa', \varkappa'', \varkappa) = \frac{2}{\pi}\left[\{\varkappa^2 - (\varkappa' - \varkappa'')^2\}\{(\varkappa' + \varkappa'')^2 - \varkappa^2\}\right]^{-\frac{1}{2}} \tag{5.3}$$

$$|\varkappa' - \varkappa''| < \varkappa < \varkappa' + \varkappa'',$$

so that there are resonances (infinite) at $\varkappa = |\varkappa' - \varkappa''|$ and $\varkappa = \varkappa' + \varkappa''$. At the end points there is the discontinuous behavior

$$\begin{cases} g_0 \to 0\,, \quad \varkappa \to (\varkappa' + \varkappa'')_+\,; \quad g_0 \to +\infty\,, \quad \varkappa \to (\varkappa' + \varkappa'')_- \\ g_0 \to 0\,, \quad \varkappa \to |\varkappa' - \varkappa''|_-\,; \quad g_0 \to +\infty\,, \quad \varkappa \to |\varkappa' - \varkappa''|_+\,. \end{cases} \tag{5.4}$$

The g_ν act like coupling constants for the purely x-dependent parts of the amplitudes. This for first order processes. For second order processes integration over the mass for internal lines must be performed, so that the overall second order coupling constant is not simply the square of any first order coupling constant as conventionally. *I.e.*, a sort of theory of « coupling constants with structure », or « matrix coupling constants » results.

Appendix I

The angular Gauss's theorem. Application to the action integral.

The angular Gauss's Theorem relates the integral of the divergence of a special kind of vector field over a solid angle to the integral of its normal com ponent over a certain curve bounding this solid angle. It is the following

$$\int_A \mathrm{d}\Omega\, \nabla_\nu V^\nu = -\int_f \mathrm{d}f_\mu V^\mu\,. \tag{A1.1}$$

In this formula V^ν is a 6-vector *homogeneous of degree* $+1$ *in* X^μ;

$$\nabla_\nu V^\nu \equiv \frac{1}{\sqrt{\mathscr{S}}}\frac{\partial}{\partial X^\nu}(\sqrt{\mathscr{S}}\, V^\nu)\,; \tag{A1.2}$$

$\mathscr{S}$ is the determinant defined in (4.3); and

$$\begin{cases} \mathrm{d}\Omega \equiv \sqrt{\mathscr{S}}\, \varepsilon_{\mu\nu\lambda\xi\varrho\tau} X^\mu\, \mathrm{d}X_1^\nu \ldots \mathrm{d}X_5^\tau\,, \\ \mathrm{d}f_\mu \equiv \sqrt{\mathscr{S}}\, \varepsilon_{\mu\nu\lambda\xi\varrho\tau} X^\nu\, \mathrm{d}\xi_1^\lambda \ldots \mathrm{d}\xi_4^\tau\,, \end{cases} \tag{A.13}$$

where the dX_i^μ $(i=1, ..., 5)$ define the surface element of *any* 5-dimensional hypersurface A subtending the solid angle Ω and the $d\xi_c^\mu$ $(c=1, ..., 4)$ define the length element of *any* 4-dimensional « closed hypercurve » bounding the solid angle Ω.

Rather than prove this theorem in 6-dimensions we shall state it in its form for ordinary 3-space and indicate the proof thereof. The theorem reads

$$\int_A d\Omega\, r^3\, \nabla\cdot\left(\frac{\boldsymbol{V}}{r^3}\right) = -\int_f \frac{\boldsymbol{r}\times d\boldsymbol{f}}{r^3}\cdot\boldsymbol{V}\,. \tag{A1.4}$$

Remember that $\boldsymbol{V}$ must be homogeneous of degree $+1$ in $\boldsymbol{r}$. A is any piece of surface subtending solid angle Ω at the origin and f is any closed curve *bounding* Ω (*i.e.*, lying in the lateral surface of the cone Ω and making a complete circuit around it). f is assumed described in the positive sense relative to the enclosed solid angle. (A1.4) is proved in two steps. 1) Apply the ordinary Gauss's Theorem to the vector field $\boldsymbol{V}/r^3$ and the volume cut out of the spherical shell $(r, r+dr)$ by the solid angle cone Ω. *The surface integrals over the two pieces of spherical surfaces cancel* due to the above property of $\boldsymbol{V}$. Divide out dr and multiply by r the resulting equation. One obtains the theorem (A1.4) for the special case: $A =$ area intercepted on the sphere r by the solid angle and $f =$ its boundary curve. 2) Notice that the left member is truly a solid angle integral, *i.e.*, independent of the particular surface A which subtends Ω and that the right member is a function of the ray only, *i.e.*, independent of the particular curve f so long as it bounds Ω, *q.e.d.*.

Consider now the variation δ of the « action » integral $I = \int d\Omega\, \mathscr{L}$. δ on any 6-scalar, spinor, tensor, etc., field $T(X)$ is defined $\delta T(X) = T'(X) - T(X)$. Therefore by definition δ will act only on Ψ and $\overline{\Psi}$ in the integral I and $[\delta, \partial/\partial X^\nu] = 0$. We have, varying and then differentiating by parts

$$\begin{aligned}
\delta\int d\Omega\, \mathscr{L} &= \int d\Omega\, \frac{4}{9}\,\delta\overline{\Psi}(R_{0\nu} - \beta_7 R)\Psi + \int d\Omega\, \frac{4}{9}\,\overline{\Psi}\,(R_{0\nu} - \beta_7 R)\,\delta\Psi = \\
&= 0 + \int \frac{d\Omega}{\sqrt{\mathscr{S}}}\,\frac{\partial}{\partial X^\nu}\{\sqrt{\mathscr{S}}\;\overline{\Psi}\,\beta_7(-\beta^{[\mu}\beta^{\nu]}X_\mu\,\delta\Psi)\} - \\
&- \int d\Omega\,\frac{\partial}{\partial X^\nu}\,\overline{\Psi}\beta_7\left\{-\beta^{[\mu}\beta^{\nu]}X_\mu - \frac{5}{2} + \frac{4}{9}\beta_7 R\right\}\delta\Psi + \\
&+ \int \frac{d\Omega}{\sqrt{\mathscr{S}}}\left(\frac{\partial}{\partial X^\nu}\sqrt{\mathscr{S}}\right)\overline{\Psi}\beta_7\beta^{[\mu}\beta^{\nu]}X_\mu\,\delta\Psi + \int d\Omega\,\overline{\Psi}\beta_7\beta^{[\mu}\beta^{\nu]}\left(\frac{\partial}{\partial X^\nu}X_\mu\right)\delta\Psi\,.
\end{aligned} \tag{A1.5}$$

In the right member of the last equation 1) the first integral, written zero, has vanished by the field equation of Ψ; 2) the third integral vanishes by the field equation on $\overline{\Psi}$; 3) the fourth integrand involves $\partial\sqrt{\mathscr{S}}/\partial X^\nu\beta^{[\mu}\beta^{\nu]}X_\mu \propto X_\nu\beta^{[\mu}\beta^{\nu]}X_\mu = 0$; 4) the fifth integrand involves $\beta^{[\mu}\beta^{\nu]}\partial X_\mu/\partial X^\nu = \beta^{[\mu}\beta^{\nu]}g_{\mu\nu} = 0$. Moreover there are no surface terms due to the change in the limits of integration when δ is due to a co-ordinate transformation $\in M$ because $\mathscr{L}$

vanishes in virtue of the field equation on Ψ. Hence only the second integral survives:

$$\text{(A1.6)} \qquad \delta\int \mathrm{d}\Omega\, \mathscr{L} = \int \mathrm{d}\Omega\, \nabla_\nu V^\nu, \qquad V^\nu \equiv -\overline{\Psi}\beta_7\beta^{[\mu}\beta^{\nu]}X_\mu\,\delta\Psi,$$

where ∇_ν has its meaning of (A1.2). V^ν is indeed homogeneous of degree $+1$ in the X^λ, therefore the angular Gauss's Theorem can be applied, giving (4.4). There, f is composed of the two parts f, f_0 corresponding to two parallel 4-dimensional surfaces in 5-space (x^m, λ) and is closed only at spatial infinity.

Appendix II

The surface element.

By the angular Gauss's Theorem, we are at liberty to take any 4-dimensional « closed hypercurve » bounding our solid angle Ω, so for convenience we take it to lie on the unit 6-sphere

$$g_{\mu\nu}X^\mu X^\nu = -1\,.$$

6-points X^μ lying on the unit sphere we shall call z^μ, thus from (2.7) we have

$$\text{(A2.1)} \qquad z^0 = \frac{1}{\lambda}\,, \qquad z^m = \frac{x^m}{\lambda}\,, \qquad z^5 = \frac{1}{2\lambda}\,(x^2+\lambda^2)\,.$$

The curves of parameter x^i $(i = 1, 2, 3)$ and λ on the unit sphere, $z^\mu = \xi_i^\mu(x^i$ $z^\mu = \xi_4^\mu(\lambda)$ respectively (i unsummed), are obtained from (A2.1) of course by letting the corresponding variable x^i or λ be the running parameter any keeping the other three variables fixed. Their length elements $\mathrm{d}\xi_i^\mu$ and $\mathrm{d}\xi$d are thus calculated to be

$$\text{(A2.2)} \qquad \begin{cases} \mathrm{d}\xi_i^0 = 0\,, \qquad \mathrm{d}\xi_i^i = \dfrac{\mathrm{d}x^i}{\lambda}\,, \qquad \mathrm{d}\xi_i^j = 0 \quad (j \neq i), \qquad \mathrm{d}\xi_i^4 = 0, \\ \mathrm{d}\xi_i^5 = \dfrac{1}{\lambda}\,x_i\,\mathrm{d}x^i \equiv z_i\,\mathrm{d}x^i \qquad (i \text{ unsummed};\ = 1, 2, 3). \\ \mathrm{d}\xi_4^0 = -\dfrac{z^0}{\lambda}\,\mathrm{d}\lambda\,, \qquad \mathrm{d}\xi_4^m = -\dfrac{z^m}{\lambda}\,\mathrm{d}\lambda \quad (m = 1, \ldots, 4), \\ \mathrm{d}\xi_4^5 = -\dfrac{1}{\lambda^2}\,(x^2+\lambda^2)\,\mathrm{d}\lambda + \mathrm{d}\lambda = -\dfrac{z^5}{\lambda}\,\mathrm{d}\lambda + \mathrm{d}\lambda\,; \end{cases}$$

thus

$$\mathrm{d}\xi_4^\mu = -\frac{z^\mu}{\lambda}\,\mathrm{d}\lambda + \delta_5^\mu\,\mathrm{d}\lambda\,.$$

These four fields $\mathrm{d}\xi_c^\mu$ define the surface element $\mathrm{d}f_\mu$ of the surface $x^4 = \text{const}$. Now since for any 5-point (x^m, λ), $z^\mu \propto X^\mu$ of course, we have from the definition (A1.3)

$$\mathrm{d}f_\mu z^\mu \equiv 0\,. \tag{A2.3}$$

Thus the first term $\propto z^\mu$ of $\mathrm{d}\xi_4^\mu$ does not contribute, and it has effectively only a 5-th component. For this reason the 5-th components of the $\mathrm{d}\xi_i^\mu$ can be disregarded and they have effectively only i-th components. Thus

$$(\mathrm{d}\xi_i^\mu)_{\text{eff}} = \frac{\mathrm{d}x^i}{\lambda}\,, \qquad (\mathrm{d}\xi_4^\mu)_{\text{eff}} = \mathrm{d}\lambda\,. \tag{A2.4}$$

Since now $-g_{\mu\nu}/g_{\lambda\xi}z^\lambda z^\xi = g_{\mu\nu}$, $\mathscr{S} = \operatorname{Det} g_{\mu\nu} = +1$, we have

$$\mathrm{d}f_\mu = \varepsilon_{\mu\nu 1235}\, z^\nu\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3 \lambda^{-3}\, \mathrm{d}\lambda\,,$$

or

$$\mathrm{d}f_0 = -x^4\,\frac{\mathrm{d}^3x\,\mathrm{d}\lambda}{\lambda^4}\,, \qquad \mathrm{d}f_4 = \frac{\mathrm{d}^3x\,\mathrm{d}\lambda}{\lambda^4}\,, \quad \text{other} \quad \mathrm{d}f_\mu = 0\,. \tag{A2.5}$$

which is the desired surface element.

It may be shown by straightforward tedious manipulation using the rules of the β-algebra, that

$$\begin{cases} X_\lambda(1+\frac{1}{2}\beta x\beta_5)\beta^{[\lambda}\beta^{\mu]}(1-\frac{1}{2}\beta x\beta_5) = \\ \mu = 0: \quad -X^0\beta_{[0}\beta_{5]}\,, \\ \mu = m: \quad -X^m\beta_{[0}\beta_{5]} - X^0\beta^m\left(\beta_0 + \beta_5\frac{\lambda^2}{2}\right), \\ \mu = 5: \quad -X^5\beta_{[0}\beta_{5]} + X^0\lambda^2\beta_{[0}\beta_{5]} + X^0\left(\beta_0 + \beta_5\frac{\lambda^2}{2}\right)\beta x\,. \end{cases} \tag{A2.6}$$

The first terms are of the form $-X^\mu\beta_{[0}\beta_{5]}$; therefore since $\mathrm{d}f_\mu X^\mu \equiv 0$, do not contribute when this expression is contracted with the surface element $\mathrm{d}f_\mu$. Thus the above expression effectively has only m-th ($m = 1, \ldots, 4$) and 5-th components. But $\mathrm{d}f_\mu$ has only 0-th and 4-th components by (A2.5). Hence

$$\mathrm{d}f_\mu\left(1+\frac{1}{2}\beta x\beta_5\right)X_\lambda\beta^{[\lambda}\beta^{\mu]}\left(1-\frac{1}{2}\beta x\beta_5\right) = -\frac{\mathrm{d}^3x\,\mathrm{d}\lambda}{\lambda^5}\,\beta^4\left(\beta_0 + \frac{\lambda^2}{2}\beta_5\right), \tag{A2.7}$$

where the substitution $X^\mu \to z^\mu$ has been made in (A2.6). This is (4.15).

RIASSUNTO (*)

Esponiamo una teoria del campo quantizzato del nucleone libero basata sulla equazione d'onda a spin $\frac{1}{2}$ proposta da Murai. Le caratteristiche notevoli sono: 1) il campo del nucleone è una rappresentazione *irriducibile* (ad 8 dimensioni) del suo gruppo fondamentale, il gruppo di 15 parametri di tutte le trasformazioni spazio-tempo che conservano gli angoli; 2) malgrado la massa non evanescente questo nucleone possiede un nuovo numero quantico β_7', la chiralità *invariante*; neutroni e protoni possono essere distinti da $\beta_7' = -1$ e $+1$ e l'operatore di carica può essere introdotto come generatore del gruppo di simmetria $\Psi' = \exp[i\alpha(\beta_7'+1)/2]\Psi$; 3) il nucleone nudo ha necessariamente uno spettro di massa; la massa è coniugata ad un altra lunghezza misurabile λ associata ad un nucleone, e si possono formare stati in cui la indefinitezza della massa nuda e di λ variano in maniera complementare. Discutiamo l'influenza di questo nuovo concetto sul procedimento fisico di misurazione della massa tramite la interazione con altri campi e sulla rinormalizzazione di questo campo nucleonico.

(*) *Traduzione a cura della Redazione.*

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Influence of the Inner Electrode on the Threshold Potential Using Siemen's Type Ozonizers under Low Frequency Silent Electric Discharge.

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Summary. — The paper reports the observations of the threshold potential V_m for the low frequency silent electric discharge using Siemen's ozonizers for air over a range of pressure from 1 to 400 mm Hg. The V_m-p curves are not linear, being concave towards the pressure axis. These results further indicate that unlike the case of the outer cylinders where small changes in the dimensions do not influence the curves appreciably, in this case small changes in the dimensions of the inner cylinders do influence the curves appreciably.

1. – Introduction.

Using metal electrodes the corona and similarity theorems have been established for air under a.c. [1] as well ad h.f. excitation [2]. If the outer metal cylinder is replaced by an externally coated glass cylinder it has been shown by Maze [3] and others [4] that at least in the Geiger region the results obtained are essentially similar and the small differences in the magnitude of the Geiger threshold and the plateau could be explained by regarding the glass walls as a serial resistance and a by-pass condenser in parallel. It was to be expected therefore that the corona relationship and the similarity theorem

[1] F. L. Jones and G. C. Williams: *Proc. Phys. Soc. (Lond.)*, **66** B, 345 (1953).
[2] F. L. Jones and G. D. Morgan: *Proc. Phys. Soc. (Lond.)*, **64** B, 560 (1951).
[3] R. Maze: *Journ. d. Phys.*, **7**, 164 (1946).
[4] M. Yasin, R. Ahmed and P. S. Gill: *Nature*, **167**, 647 (1951).

would hold good for Maze type counters also if an allowance is made for the fall of potential on the glass walls (5). The results reported earlier are in accord with this. However, in an ozonizer, the central axial wire is also replaced by a glass cylinder of diameter comparable with that of the outer cylinder. This leads to a change in the nature of the electrodes as well as in the asymmetry. No data in respect of this are available in literature. It was therefore of interest to investigate the influence of the inner electrode on the threshold potential when the diameter of the inner cylinder varies within wide limits. Such studies would be very useful for the data on Joshi effect (6) which have been investigated mostly using all glass ozonizers.

2. – Experimental.

The general experimental arrangement and procedure were essentially the same as those reported earlier (7). The air used was carefully filtered through glass wool to remove dust particles andw as stored over phosphorus pentoxide for twentyfour hours. All glass Siemen's ozonizers were used. The dimensions of the ozonizers are recorded in Table I. To investigate the possible

TABLE I. – *Effective length of the tubes* — 8.00 cm.

Tube No.	Dimensions of the tubes			
	Outer diameter of the outer tube cm	Inner diameter of the outer tube cm	Outer diameter of the inner tube cm	Inner diameter of the inner tube cm
1	2.63	2.30	0.64	0.16
2	2.63	2.30	0.69	0.45
3	2.63	2.30	1.41	1.21
4	2.63	2.30	2.03	1.82
5	3.30	3.10	1.20	1.00
6	3.30	3.10	2.30	2.10
7	1.75	1.60	1.20	1.0
8	1.75	1.60	0.70	0.5

influence of a change in the inner cylinder with the same outer cylinder, different tubes were used. To investigate the possible role of the thickness of the glass wall a capillary tube was used as the inner cylinder for tube 1, while the outer glass cylinder was the same as that for tubes 2 to 4. A solution of

(5) D. P. JATAR and H. D. SHARMA: *Curr. Sci.*, **25**, 324 (1956); H. D. SHARMA: *Res. Bull. Saugar University Phys. Soc.*, **1**, 51 (1957).

(6) S. S. JOSHI: *Curr. Sci.*, **8**, 548 (1939).

(7) D. P. JATAR and H. D. SHARMA: *Journ. Sci. Industr. Res.*, **15** B, 417 (1956); D. P. JATAR and H. D. SHARMA: *Nature*, **178**, 1073 (1956).

sodium chloride in the inner tube of the full ozonizer and a helical of copper wire wound tightly round the outer tube served as the two electrodes. The ozonizers were initially cleaned with chromic acid and then with distilled water. They were then dried and baked at 200 °C in vacuum to remove wall absorbed impurities and were excited by an H.T. transformer using 50 kHz a.c. The discharge current flowing through the tube was measured by a moving coil galvanometer in the plate circuit of a detector (1H 5GT) connected across a serial carbon resistance, 10 000 ohms, in the L.T. line. Before taking any observations the discharge tubes were aged for at least one hour at every pressure under a discharge well above the threshold potential V_m.

3. – Results and discussion.

The threshold potential, V_m, (kV, a.c. peak value) was indicated by a sudden rise in the current flowing through the system, the initiation of a glow characteristic of the gas and the appearance of pulses on the current oscillogram.

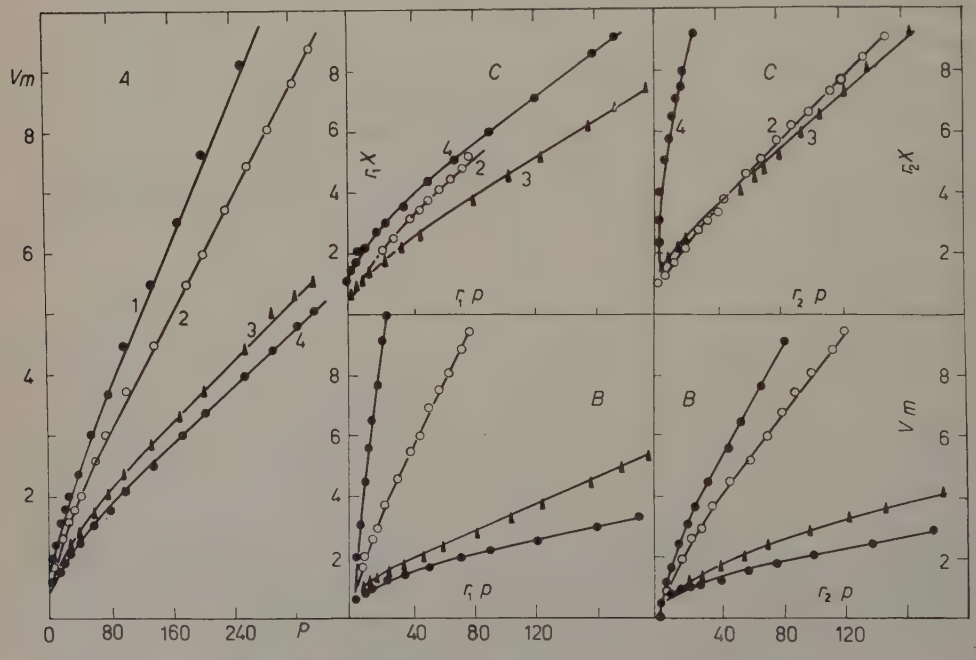


Fig. 1*a*). - Variation of the threshold potential, V_m, with the pressure of the gas. Numbers on curves correspond to the numbers of the tubes in Table I. - *b*) $V_m - r_1p$ and $V_m - r_2p$ curves for the same tubes. $2r_1$ and $2r_2$ being the inner and outer diameters of the inner glass cylinder. - *c*) i) $r_1X - r_1p$ curves for the same tubes. Using the inner diameter of the inner glass cylinder and the outer diameter of the outer glass cylinder for the calculation of r_1X; ii) $r_2X - r_2p$ curves for the same tubes. Using the outer diameter of the outer glass cylinder for the calculation of r_2X.

Values of V_m over a pressure range from 1 to 500 mm Hg for discharges between coaxial glass cylinders are shown in Figs. 1*A*, 2*A* and 3*A* for ozonizers indicated on the curves. It is seen that the curves are sensibly straight lines except at low pressures where they are concave towards the pressure axis.

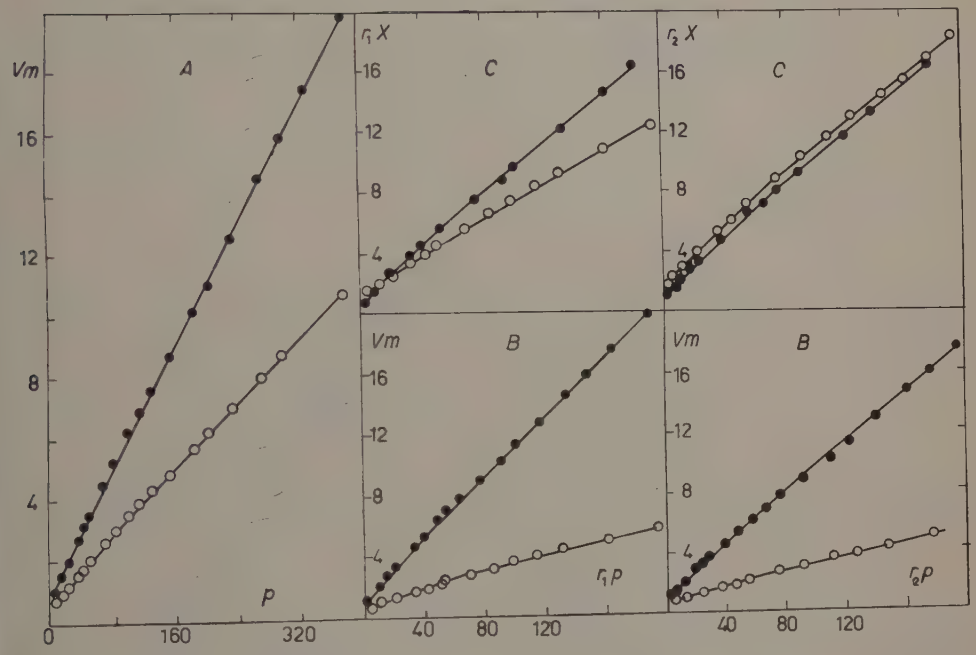


Fig. 2*a*). - $V_m - p$ curves for tubes 5 and 6 in Table I. - *b*) $V_m - r_1 p$ and $V_m - r_2 p$ curves for the same tubes. - *c*) $r_1 X - r_1 p$ and $r_2 X - r_2 p$ curves for the same tubes.

It is of interest to note here that for a given outer cylinder an increase in the radius of the inner cylinder causing a decrease in the air gap, decreases V_m. A comparison of V_m for tube 1 with that for tube 2, having dimensions practically equal to those of 1 except for the wall thickness of the inner electrode, shows that V_m for the former capillary tube is appreciably larger indicating that there is a larger fall in potential on the glass walls corresponding to the capillary used.

The variation of V_m with $r_1 p$ and $r_2 p$ (where r_1 and r_2 are the inner and outer radii of the inner tube in cm) is shown in Figs. 1*B*, 2*B* and 3*B*. The curves are sensibly linear with a small departure at low rp values. The $rX - rp$ curves for these tubes are plotted in Figs. 1*C*, 2*C* and 3*C*, where X is the field, in volts per cm. The variation of X/p for tubes 7 and 8 is also plotted in Fig. 3*C*. It is interesting to note that the curves for different tubes do not coincide.

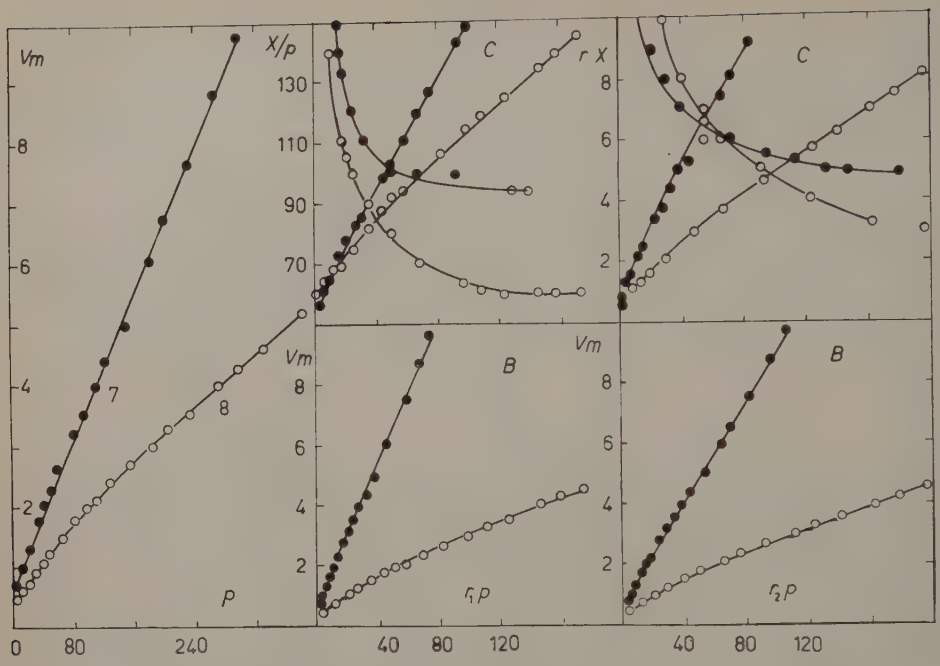
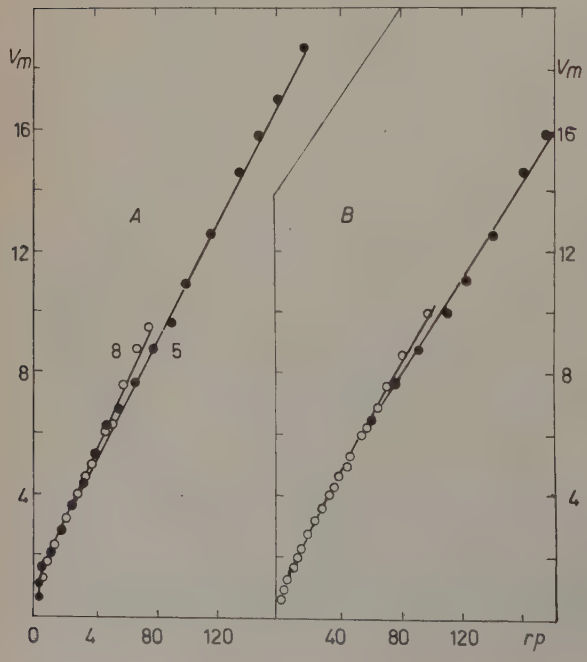


Fig. 3*a*). – $V_m - p$ curves for tubes 7 and 8 in the Table I. - *b*) $V_m - r_1 p$ and $V_m - r_2 p$ curves for the same tubes. - *c*) $r_1 X - r_1 p$ and $r_2 X - r_2 p$ curves for the same tubes. $X/p - r_1 p$ and $X/p - r_2 p$ curves for the same tubes.



The results for other tubes are similar. Moreover when similar tubes are considered namely 5 and 8, the $V_m - rp$, $rX - rp$ and $X/p - rp$ curves coincide within the experimental error except slight departure for high rp values cf. Fig. 4 and 5, indicating that the similarity theorem holds good in this type of discharge also.

Fig. 4*a*). – $V_m - r_1 p$ curves for tubes 5 and 8 in Table I. - *b*) $V_m - r_2 p$ curves for the same tubes.

The analogy between the results reported here and those reported earlier for a semi-ozonizer ([8]) suggests that the mechanism of breakdown in the two cases should be similar. Thus the nature of the variation of V_m with the pressure of the gas is similar to that reported earlier and has already been discussed. In the present case also, the influence of the electrode diameter may be interpreted in terms of a change in the corresponding air gap, a decrease in the latter leading to a decrease in the threshold potential.

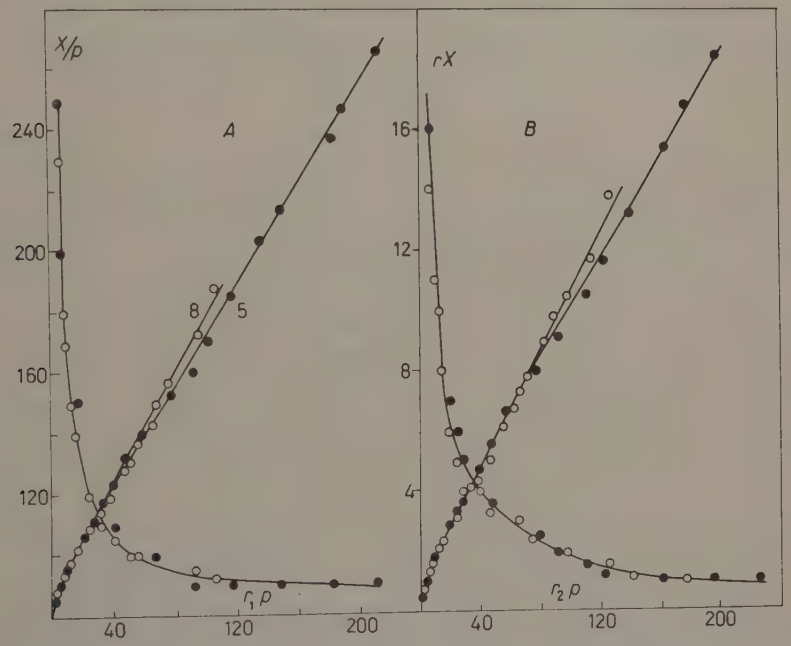


Fig. 5*a*). – $X/p - r_1 p$ and $r_1 X - r_1 p$ curves for tubes 5 and 8 in Table I. - *b*) $X/p - r_2 p$ and $r_2 X - r_2 p$ curves for the same tubes.

Detailed studies ([9]) using coaxial cylinder coronas have shown the possible importance of the central electrode dimensions. Keeping this sensibly constant, the dependence of V_m, on the outer cylinder diameter has been investigated experimentally as well as theoretically ([10]). No efforts seem to have been made, however, to investigate the possible method of comparison

([8]) H. D. Sharma: *Acta. Phys. Pol.*, **17**, 183 (1958).

([9]) J. S. Townsend: *Electrons in Gases*, Hutchinson's Scientific Publications (1947); Huxley: *Phil. Mag.*, **5**, 721 (1928).

([10]) L. B. Loeb: *Fundamental Processes of Electrical Discharges in Gases* (New York, 1938).

of data when the diameter of the inner cylinder varies within wide limits. Figs. 1, 2 and 3 show the data for eight tubes. For tubes 1, 2, 3 and 4, the outer cylinder is the same and the inner cylinder is changed from 0.16 to 1.82 cm. For tubes 5 and 6, the outer cylinder is the same and the inner is changed, similarly for tubes 7 and 8. It is clear from the Fig. 3*C* that the $rX - rp$ and $X/p - rp$ curves for these tubes do not coincide. The same results are obtained with tubes 1 to 6 though the values of $X/p - rp$ are not plotted here. These results indicate that unlike the case of the outer cylinder where small changes in the dimensions do not influence the $rX - rp$ curves appreciably, in this case small changes in the dimensions of the inner cylinder do influence the $rX - rp$ curves appreciably.

* * *

My grateful thanks are due to Dr. D. P. Jatar for his helpful criticism throughout the progress of this work.

RIASSUNTO (*)

L'articolo riferisce le osservazioni del potenziale di soglia V_m per la scarica elettrica silenziosa a bassa frequenza nell'uso degli ozonizzatori di aria Siemens nel campo di pressioni fra 1 e 400 mm Hg. Le curve V_m-p non sono lineari, ma presentano una concavità verso l'asse della pressione. Questi risultati indicano inoltre che diversamente dal caso dei cilindri esterni le cui piccole variazioni di dimensioni non influenzano in modo apprezzabile le curve, nel caso dei cilindri interni invece piccole variazioni delle dimensioni hanno un'apprezzabile influenza sulle curve.

(*) *Traduzione a cura della Redazione.*

Baryon Mass-Differences and Symmetries of Strong Interactions.

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(ricevuto il 7 Gennaio 1960)

Summary. — The present paper gives a scheme of the strong interactions of the baryons, which from the start takes account of the different baryon masses, and yet has a high degree of symmetry without forbidding any of the known fast reactions. Only the usual fermion-fermion-boson type of interactions are employed. We start from the results already obtained in the papers *A* and *B* (1), which made use of the doublet approximation where the baryons are grouped into four isobaric doublets having the same mass. The four doublets are described by a single 32-component spinor and their interactions written using 32×32-matrices operating on this spinor. In *A* and *B* the K-interaction, neglecting the mass-differences, was taken to be an invariant under rotations in a 4-dimensional Euclidean space called the *hypercharge space*, the K-mesons being described by four real fields considered as the components of a vector in this space. The π-interaction was, of course, invariant under rotations in the usual 3-dimensional isobaric spin space. Significant for the present paper are the six generators of infinitesimal rotations in the hypercharge space, grouped here into two sets of quantities Y_i and Z_i $(i=1, 2, 3)$, which generate infinitesimal rotations in two different 3-dimensional subspaces of the hypercharge space, which are called, respectively, the *hypercharge-spin space* and the *hypernumber-spin space* (Y_i are called the hypercharge-spin operators and Z_i the hypernumber-spin operators). In *A* the $\mathcal{N}$-Ξ mass-difference was introduced by adding to the K-interaction another term so that the resulting Lagrangian was no longer invariant under rotations in the hypercharge-spin space. This interaction is invariant only under rotations about the 3-axis of this space (this expresses the conservation of the hypercharge or the strangeness) and under any rotation in the hypernumber-spin space. In the present paper, the final step of introducing the Λ-Σ mass-splitting is carried through. This is done by taking for the final π- and K-interaction Lagrangian a linear combi-

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nation of the original one of A and the one obtained therefrom by interchanging the roles of the fictitious particles Y^0 and Z^0 ($Y^0=(\Sigma^0+\Lambda^0)/\sqrt{2}$, $Z^0=(\Sigma^0-\Lambda^0)/\sqrt{2}$). Then only *four* independent constants (F, F', g and b) enter the theory. It turns out that the various terms of the final Lagrangian have exactly the same form as the terms of the d'Espagnat-Prentki Lagrangian. The difference is that the eight coupling constants of the latter are now *not* independent, but are expressed in terms of only four parameters F, F', g and b. Our Lagrangian now conserves neither the total hypernumber-spin nor the total isobaric spin of the doublet approximation, but it does conserve the quantity obtained as the vector sum of the two. This quantity we call the « *effective isobaric spin* ». It has exactly the same values for the different particles as the ones given by the Gell-Mann-Nishijima scheme. This leads us to identify the isobaric spin space with the hypernumber-spin space, and thus consider it as a subspace of the 4-dimensional hypercharge space. We find further that our theory is invariant under boson-, spinor- and charge-conjugation [12] separately. Finally it is seen that the charge-state symmetries inherent in our theory are such that, assuming CP-invariance, they automatically lead to P-invariance.

Introduction.

The strong interactions of the baryons with the mesons and the charge-state symmetries underlying them have been studied from rather different points of view during the last few years [1]. As one of the earlier attempts in this direction, SCHWINGER [2] gave a theory, which, on the one hand, presented a scheme of the baryon charge-states, and, on the other, introduced a direct KKπ-interaction in order to explain the baryon mass differences. Somewhat later, GELL-MANN [3] adopted the point of view that, as a first step, one may neglect all the baryon mass-differences thereby obtaining the highest degree of symmetry, which is later reduced by secondary perturbations. He replaced the isobaric spin singlet Λ^0 and the triplet Σ by two doublets. The four baryon doublets thus obtained (including the $\mathcal{N}$ and the Ξ doublets) were assumed to have exactly the same strong interaction with the π-mesons. This is the well-known global symmetry. In this scheme the K-interactions play only a secondary role. An extension of this scheme employing these four baryon doublets to the K-interactions also was given independently by

(1) B. D'ESPAGNAT and J. PRENTKI: *Nucl. Phys.*, **1**, 33 (1956); A. SALAM: *Nucl. Phys.*, **2**, 173 (1956); B. D'ESPAGNAT, J. PRENTKI and A. SALAM: *Nucl. Phys.*, **3**, 446 (1957); A. SALAM and J. C. POLKINGHORNE: *Nuovo Cimento*, **4**, 848 (1955).

(2) J. SCHWINGER: *Phys. Rev.*, **104**, 1164 (1956); *Ann. Phys.*, **2**, 407 (1957).

(3) M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957).

TIOMNO (4) and by one of the present authors (N.D.) (5). This is usually referred to as the « doublet approximation ». PAIS (6), however, quite conclusively proved the incompatibility of this highly symmetrical scheme with experiments. He showed that many reactions are forbidden in this scheme while they are actually observed to take place with cross-sections comparable with those of the allowed ones. An example of such a forbidden reaction is the K^+-n charge exchange scattering. This work of PAIS marked a turning point, as it became clear that the questions of the symmetries and their perturbations cannot be separated. One then turned to look for suitable changes in the scheme. These were either located in the main body of the interactions, by, for example, taking the relative parities of some of the baryons odd or they were caused due to the addition of a new type of interaction like the direct $\bar{K}K\pi\pi$- (7) or the $\bar{K}^+K^0\pi^+$-interaction (8) (the latter, assuming parity conservation, is only possible if K^+ and K^0 have opposite parities, to which there is no objection provided that all the K particles are taken to be isobaric singlets).

Thus, while the initial highly symmetrical approaches were characterized by a minimum number of coupling constants (two in the case of strongest symmetry, one each for the π- and the K-interactions, neglecting all mass differences and using the same type of coupling), the new approach with the additional perturbations required the introduction of more interaction constants in order to improve agreement with the experimental data. However these perturbations turned out to be so important, and their effect so irregular in order to explain the data, that the original symmetries were almost completely obliterated so that one could hardly speak of them.

This situation is, from a general theoretical point of view, so unsatisfactory that one can hardly refrain from believing that something important has been missed and that it is worthwhile reconsidering the whole problem. The question is whether any interaction scheme can be found, which in spite of retaining a good many of the original symmetries, reproduces the baryon mass-differences and does not forbid any of the observed reactions.

The present paper, which continues and concludes a preceding group of researches (9), gives a positive answer to this question. It will be shown, that without introducing any interaction other than of the usual fermion-fermion-

(4) J. TIOMNO: *Nuovo Cimento*, **6**, 69 (1957).

(5) N. DALLAPORTA: Proceedings of the Conference on Mesons and recently discovered particles V, 3 (Padua-Venice 1957); *Nuovo Cimento*, **7**, 200 (1958).

(6) A. PAIS: *Phys. Rev.*, **110**, 574 (1958).

(7) S. BARSHAY: *Phys. Rev.*, **109**, 2160 (1958).

(8) A. PAIS: *Phys. Rev.*, **112**, 624 (1958).

(9) N. DALLAPORTA and T. TOYODA: *Nuovo Cimento*, **14**, 142 (1959), here referred to as *A*; see also *Nuovo Cimento*, **12**, 539 (1959); N. DALLAPORTA and V. DE SANTIS: *Nuovo Cimento*, **14**, 225 (1959), here referred to as *B*.

boson type, we may give a scheme in which the $\mathcal{N}$, Ξ, Λ and Σ masses automatically differ in the desired manner, while at the same time a good amount of symmetry is still preserved without forbidding any of the observed fast reactions. This is essentially achieved by considering the K-interactions as conditioned by certain rotational properties in a 4-dimensional Euclidean « *hypercharge-space* ». The idea of using such a space is, of course, not new. It has been employed by PAIS ([8,10]) in some of his earliest attempts to group the strong π- and K-interactions according to a 4-dimensional isospin formalism. Here, instead, the 4-dimensional space is used only for the K-interactions in a manner quite similar to that of a model first proposed by TIOMNO ([4]) and which, in the case of the doublet approximation, has already been developed in the two preceding papers *A* and *B* ([9]). The most important feature of this approach is the use made of the two sets of 3-dimensional generators of infinitesimal rotations in the 4-dimensional « hypercharge space ». One of these, $\boldsymbol{Y}$, has been called in *B* as the *hypercharge-spin* (Y_3 is half the hypercharge), while the other, $\boldsymbol{Z}$, called the *hypernumber-spin* will be interpreted later in the following.

As the investigation of the present paper depends on the results already obtained in *A* and *B*, we shall begin by giving in Section **1** a short resumé of these papers, at the same time introducing some convenient changes and removing some inessential errors left therein. The main result at this stage is the hypercharge *independence* of the K-interactions (and, of course, the usual charge independence of the π-interactions). In Section **2** we shall recapitulate, following *A*, how the mass-differences of the $\mathcal{N}$ and the Ξ is introduced in the doublet approximation breaking thereby the hypercharge independence. In Section **3** we proceed to the next step, which is that of getting over the doublet approximation. A mechanism is introduced which leads to the Λ-Σ mass-splitting and we obtain, instead of the two hyperon (*) doublets introduced by GELL-MANN, again a singlet Λ and a triplet Σ. This is achieved by writing the final interaction Lagrangian density as a linear combination of the one in Section **2** and the one obtained from it by interchanging the roles of the Y^0 and the Z^0. This gives rise to a constant of the motion, which we interpret as the *effective isospin*. This isospin coincides with the assignments given by the Gell-Mann–Nishijima scheme ([11]). It arises from the addition of

([10]) A. PAIS: Proceedings of the Rochester Conference (1955); *Proc. Nat. Acad. Sc.*, **40**, 9, 835 (1954).

(*) Throughout this paper we use the name *hyperon* for the Λ and the Σ while the other baryons are referred to as the nucleon ($\mathcal{N}$) and the Ξ. The hyperon doublets introduced by Gell-Mann are $\begin{pmatrix}\Sigma^+\\ Y^0\end{pmatrix}$ and $\begin{pmatrix}Z^0\\ \Sigma^-\end{pmatrix}$, where $Y^0 = \dfrac{\Sigma^0 + \Lambda^0}{\sqrt{2}}$ and $Z^0 = \dfrac{\Sigma^0 - \Lambda^0}{\sqrt{2}}$.

([11]) M. GELL-MANN: *Phys. Rev.*, **92**, 833 (1953); T. NAKANO and K. NISHIJIMA: *Progr. Theor. Phys.*, **10**, 581 (1953); K. NISHIJIMA: *Prog. Theor. Phys.*, **12**, 107 (1954).

the doublet approximation isospin and the hypernumber-spin **Z**. In our final Lagrangian density the various terms have the same form as the terms of the d'Espagnat-Prentki ([1]) scheme. The difference is that only *four* independent constants enter our theory. In the next Section **4**, some further invariance properties of the scheme will be considered. It will be seen that the theory is invariant under the $\mathscr{B}$ (boson), $\mathscr{S}$ (spinor) and $\mathscr{C}$ (charge) conjugations ([12]) separately, as was also the case in the doublet approximations of Sections **1, 2** (cf. *A*). We shall, moreover, discuss in this section the relations between the eight coupling constants of the d'Espagnat-Prentki Lagrangian according to our scheme. Section **5** will deal with the question of a possible relation between the charge-state symmetries and parity conservation in this scheme. It will be shown that if we assume the Lagrangian density to contain a parity violating mixture of a scalar and a pseudoscalar term and no derivative couplings, then by demanding *CP*-invariance we obtain, on account of the strong charge-state symmetries of the theory, the vanishing of either the scalar term or the pseudoscalar term. The theory thus automatically conserves *C* and *P* separately; and here this naturally includes also the Λ-interactions. Thus if the detailed predictions of our theory were found to be in agreement with experimental results, we shall be able to conclude that parity conservation does not have to be considered as a separate requirement for the strong interactions, but that it follows from the inherent charge symmetries and *CP*-invariance.

1. – Let us consider the interaction scheme in the doublet approximation as already developed in *A* and *B* (*). The results are best visualized with the help of the diagram shown in Fig. 1. In the above papers the different observed baryons are considered as different states of the same fermion (called the *Baryon*) described by a 32-component spinor. It is assumed that all the baryons have the

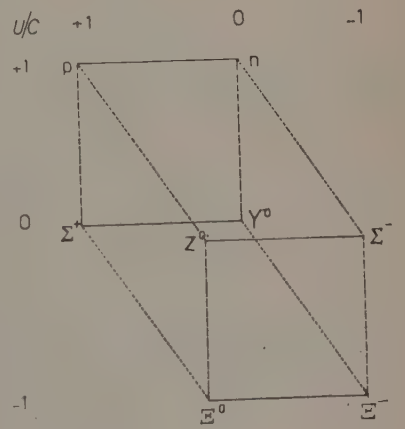

Fig. 1. – The unbroken lines represent the π-transition, the broken lines the transitions involving the neutral K mesons, and the dotted lines those involving the charged K mesons. *C* is the charge and *U* the hypercharge of the various baryons.

([12]) P. Budini, N. Dallaporta and L. Fonda: *Nuovo Cimento*, **9**, 316 (1958).

(*) In the present paper the symbols used are in some cases different from those used in *A* and *B* in order to make the notation clearer.

same parity. We have considered this spinor variously in three different representations:

i) the *completely mixed* representation $X^{(32)}$, ii) the *partially separated* or *intermediate* representation $\Psi^{(32)}$ and iii) the *completely separated* representation $\psi^{(32)}$.

i) In the completely mixed representation the « Baryon » spinor $X^{(32)}$ satisfies the free-field equation

$$\left(G_\mu \frac{\partial}{\partial x_\mu} + m\right) X^{(32)} = 0 \,, \tag{1}$$

where the repeated greek index μ, signifying the four space-time components, is summed over 1, 2, 3, 4 (we take $\hbar = c = 1$, $x_4 = it$). We may write

$$X^{(32)} = \begin{bmatrix} X_a^{(16)} \\ X_b^{(16)} \end{bmatrix}, \tag{2}$$

where the group of 16-components indexed a represents a mixture of the Λ^0 and the Ξ; the group indexed b is a mixture ot the hyperons (Σ^+, Υ^0, Z^0, Σ^-). The G_μ are 32×32 matrices:

$$G_\mu = \left[\begin{array}{cc|cc} 0 & -\Gamma_\mu & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \Gamma_\mu & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -\Gamma_\mu \\ & & \Gamma_\mu & 0 \end{array}\right]; \quad \Gamma_\mu = \begin{bmatrix} 0 & -\gamma_\mu \\ \gamma_\mu & 0 \end{bmatrix}; \tag{3}$$

$$\gamma_4 = \beta \,; \quad \gamma_k = -i\beta\alpha_k \ (k = 1, 2, 3) \,; \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \,;$$

where β and α_k are the well-known (4×4)-Dirac matrices. We shall also define

$$G_5 = \left[\begin{array}{cc|cc} \Gamma_5 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \Gamma_5 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \Gamma_5 & 0 \\ & & 0 & \Gamma_5 \end{array}\right]; \quad \Gamma_5 = \begin{bmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{bmatrix}. \tag{4}$$

The G-matrices have the following properties:

$$\begin{cases} G_\mu^\dagger = G_\mu \,, \quad G_5^\dagger = G_5 \,, \quad G_5^2 = I^{(32)} \,; \\ G_\mu G_\nu + G_\nu G_\mu = 2\delta_{\mu\nu} I^{(32)} \,; \\ G_\mu G_5 + G_5 G_\mu = 0 \,. \end{cases} \tag{5}$$

$I^{(32)}$ means the (32×32)-unit matrix.

ii) By means of the unitary (also hermitian) transformation

$$X^{(32)} = S\Psi^{(32)}, \tag{6}$$

where

$$S = S^{\dagger} = S^{-1} = \left[\begin{array}{cc|cc} B & A & & \\ A & -B & \multicolumn{2}{c}{0} \\ \hline & & B & A \\ \multicolumn{2}{c|}{0} & A & -B \end{array}\right]; \quad A = \frac{1+\Gamma_5}{2}, \quad B = \frac{1-\Gamma_5}{2}, \tag{7}$$

we get at the intermediate or the partially separated representation in which the « Baryon » is described by the spinor

$$\Psi^{(32)} = \begin{bmatrix} \Psi_a^{(8)} \\ \Psi_b^{(8)} \\ \Psi_c^{(8)} \\ \Psi_d^{(8)} \end{bmatrix}. \tag{8}$$

Here the 8-component spinors indexed a, b, c, and d represent, respectively, the mixtures of p and n, Ξ^0 and Ξ^-, Σ^+ and Y^0, and Z^0 and Σ^-.

iii) Finally by a further unitary (also hermitian) transformation

$$\left\{\begin{array}{l} \Psi^{(32)} = \left[\begin{array}{cc|cc} S & 0 & & \\ 0 & S & \multicolumn{2}{c}{0} \\ \hline & & S & 0 \\ \multicolumn{2}{c|}{0} & 0 & S \end{array}\right] \psi^{(32)}; \\ S = \begin{bmatrix} \beta & \alpha \\ \alpha & -\beta \end{bmatrix}, \quad \alpha = \dfrac{1+\gamma_5}{2}, \quad \beta = \dfrac{1-\gamma_5}{2}; \end{array}\right. \tag{9}$$

we get at the completely separated representation where

$$\psi^{(32)} = \begin{bmatrix} \psi_{\mathrm{p}} \\ \psi_{\mathrm{n}} \\ \psi_{\Xi^0} \\ \psi_{\Xi^-} \\ \psi_{\Sigma^+} \\ \psi_{Y^0} \\ \psi_{Z^0} \\ \psi_{\Sigma^-} \end{bmatrix}. \tag{10}$$

The ψ_{p}, ψ_{n}, ψ_{Ξ^0} etc., are now the ordinary 4-component Dirac spinors for the p, n, Ξ^0 etc., respectively. Each one satisfies the ordinary Dirac equation.

The names employed for the three representations should be clear from the foregoing.

We would like to stress here that the different mixed representations, X, Ψ, were introduced in A for convenience in discussing « γ_5-invariance » properties of the Baryon-field equation. It is, however, not necessary for us to go to the mixed representations for discussing the charge-state symmetries. We have introduced them again here only for the sake of consistency and to avoid any confusion.

According to A, we now write the Lagrangian density in the X-representation (we shall often from now on drop the index-32 on the spinor) as a scalar in space-time, in the following manner:

$$\mathscr{L} = \frac{1}{2}\left\{\bar{X}G_\mu \frac{\partial X}{\partial x_\mu} - \frac{\partial \bar{X}}{\partial x_\mu} G_\mu X\right\} + m\bar{X}X + \tag{11}$$

$$+ \frac{ie}{2}\bar{X}(G'_\mu + G''_\mu)XA_\mu + iF\sum_{K=1}^{4}\bar{X}\Omega_K G_5^{K+1}X\varphi_K + ig\sum_{j=1}^{3}\bar{X}T_j G_5^{j+1}X\Pi_j\,,$$

where

$$\bar{X} = X^\dagger G_4\,.$$

m is the bare Baryon-mass, A_μ is the electromagnetic 4-potential, φ_K ($K = 1, \ldots, 4$) are four real fields describing the K-mesons which are assumed all to have the same parity; the φ_K are taken to be the components of a vector in the 4-dimensional hypercharge space. Π_j ($j = 1, 2, 3$) are the real components of a vector in the 3-dimensional isobaric spin space, and they describe the π-mesons. Both the Π_j and the φ_K are here considered as pseudoscalars in space-time. If φ_K were scalar we have only to take G_5^K instead of G_5^{K+1} in the φ_K-interaction term in (11). e, F and g are, respectively, the electromagnetic K- and π-coupling constants. The various operators occuring in (11) are represented by the following 32×32 matrices:

$$\left\{\begin{array}{l} G'_\mu = \left[\begin{array}{cc|cc} 0 & \Gamma_5\Gamma_\mu & & \\ \Gamma_5\Gamma_\mu & 0 & & 0 \\ \hline & & 0 & \Gamma_5\Gamma_\mu \\ & 0 & \Gamma_5\Gamma_\mu & 0 \end{array}\right], \\[2ex] G''_\mu = \left[\begin{array}{cc|cc} 0 & -\Gamma_5\Gamma'_\mu & & \\ \Gamma_5\Gamma'_\mu & 0 & & 0 \\ \hline & & 0 & -\Gamma_5\Gamma'_\mu \\ & 0 & \Gamma_5\Gamma'_\mu & 0 \end{array}\right], \\[2ex] \Gamma'_\mu = \left[\begin{array}{cc} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{array}\right]; \end{array}\right. \tag{12}$$

$$
\Omega_1 = i \left[\begin{array}{cc|cc} & & 0 & I^{(8)} \\ & 0 & I^{(8)} & 0 \\ \hline 0 & -I^{(8)} & & \\ -I^{(8)} & 0 & & 0 \end{array}\right],
$$

$$
\Omega_2 = \left[\begin{array}{cc|cc} & & 0 & -I^{(8)} \\ & 0 & I^{(8)} & 0 \\ \hline 0 & I^{(8)} & & \\ -I^{(8)} & 0 & & 0 \end{array}\right],
$$

$$
\Omega_3 = i \left[\begin{array}{cc|cc} & & -I^{(8)} & 0 \\ & 0 & 0 & I^{(8)} \\ \hline I^{(8)} & 0 & & \\ 0 & -I^{(8)} & & 0 \end{array}\right],
$$

$$
\Omega_4 = \left[\begin{array}{cc|cc} & & I^{(8)} & 0 \\ & 0 & 0 & I^{(8)} \\ \hline I^{(8)} & 0 & & \\ 0 & I^{(8)} & & 0 \end{array}\right],
$$

$$
\Omega_5 = \Omega_1\Omega_2\Omega_3\Omega_4 = \left[\begin{array}{cc|cc} I^{(8)} & 0 & & \\ 0 & I^{(8)} & & 0 \\ \hline & & -I^{(8)} & 0 \\ & 0 & 0 & -I^{(8)} \end{array}\right],
$$

(13)

$$
\Omega_K\Omega_L + \Omega_L\Omega_K = 2\delta_{KL} I^{(32)}, \qquad (K, L = 1, \ldots, 4),
$$

$$
\Omega_5\Omega_K + \Omega_K\Omega_5 = 0,
$$

$$
G_\mu\Omega_K + \Omega_K G_\mu = 0, \qquad (K = 1, 3)
$$

$$
G_\mu\Omega_K - \Omega_K G_\mu = 0, \qquad (K = 2, 4)
$$

$$
G_5\Omega_K - \Omega_K G_5 = 0;
$$

$$
(14)\quad
\begin{cases}
T_j = \left[\begin{array}{cc|cc} \tau_j' & 0 & & \\ 0 & \tau_j' & \multicolumn{2}{c}{0} \\ \hline & & \tau_j' & 0 \\ \multicolumn{2}{c|}{0} & 0 & \tau_j' \end{array}\right], & (j=1,2,3), \\
\tau_1' = \begin{bmatrix} 0 & I^{(4)} \\ I^{(4)} & 0 \end{bmatrix}, \quad \tau_2' = \begin{bmatrix} 0 & iI^{(4)} \\ -iI^{(4)} & 0 \end{bmatrix}, \quad \tau_3' = \begin{bmatrix} -I^{(4)} & 0 \\ 0 & I^{(4)} \end{bmatrix}, & \\
\left.\begin{array}{l} [T_i, T_j] \equiv T_i T_j - T_j T_i = 2iT_k\,, \\ T_i T_j = -T_j T_i = iT_k\,. \end{array}\right\} & (i,j,k = \text{cycl. } 1,2,3);
\end{cases}
$$

$$
(14a)\quad
\begin{cases}
[\Omega_K, T_j] = 0\,, & \\
G_\mu T_i + T_i G_\mu = 0\,, & (i=1,3) \\
G_\mu T_i \quad T_i G_\mu = 0\,, & (i=2) \\
G_5 T_i - T_i G_5 = 0\,. &
\end{cases}
$$

We should like to emphasize here that the G_μ, G_μ' and G_μ'' operate essentially on the space-time co-ordinates of the spinor, while the Ω_K operate on the 4-dimensional hypercharge co-ordinates. $\bar{X}\Omega_K X$ ($K = 1, \ldots, 4$) are the components of a vector in the 4-dimensional hypercharge space. The T_j operate on the usual isobaric spin co-ordinates of the baryon state. The last term in (11), representing the π-interactions, is invariant under rotations in the 3-dimensional isobaric spin space. This leads to the charge independence of the π-interactions and so to the conservation of the total isobaric spin $\frac{1}{2}\boldsymbol{T} + \boldsymbol{T}^{(\pi)}$ in the doublet approximation. The three components of $\boldsymbol{T}^{(\pi)}$ are the isobaric spin operators for the π.

The K-interaction term in (11) is clearly invariant under rotations in the 4-dimensional hypercharge space. To understand the properties of this term it is useful to go to the partially or the completely separated representations Ψ or ψ. This is because in these representations each of the separate components $\Psi_\alpha^{(8)}, \ldots$ or $\psi_p, \ldots$ have a sharply defined value for the « *strangeness* » or the « *hypercharge* ». Let us then write the K-interaction term in the Ψ-representation using the transformation (6) as follows:

$$
(15)\qquad \mathscr{L}_K = iF \sum_{L=1}^{4} \bar{\Psi} \omega_L \Psi \varphi_L\,,
$$

where it turns out that the relevant transformed operators are represented by the matrices:

$$
(16a)\qquad \omega_1 = \Omega_1 \Gamma_5\,, \quad \omega_2 = -\Omega_2 \Gamma_5\,, \quad \omega_3 = -\Omega_3 \Gamma_5\,, \quad \omega_4 = \Omega_4 \Gamma_5\,,
$$

where, for example, we mean that

$$\omega_1 = i \left[\begin{array}{cc|cc} & 0 & 0 & \Gamma_5 \\ 0 & & \Gamma_5 & 0 \\ \hline 0 & -\Gamma_5 & & \\ -\Gamma_5 & 0 & 0 & \end{array}\right]. \tag{16b}$$

Following B and the general formalism of PAIS ([10]), we may now construct the 6 generators of infinitesimal rotations in the 4-dimensional hypercharge space. Let us define

$$M_{LK} = -\frac{i}{4}\,[\omega_L, \omega_K], \qquad (L, K = 1, 2, ..., 4), \tag{17}$$

and derive from these the following two sets of 3-dimensional generators $\boldsymbol{Y}$ and $\boldsymbol{Z}$:

$$\left\{\begin{array}{l}
Y_1 = \dfrac{1}{2}\,(M_{23} + M_{14}) = \dfrac{1}{2}\left[\begin{array}{cc|c} 0 & I^{(8)} & \\ I^{(8)} & 0 & 0 \\ \hline & 0 & 0 \end{array}\right]; \\[2ex]
Z_1 = \dfrac{1}{2}\,(M_{23} - M_{14}) = \dfrac{1}{2}\left[\begin{array}{cc|cc} & & & \\ & 0 & & 0 \\ \hline & & 0 & I^{(8)} \\ & 0 & I^{(8)} & 0 \end{array}\right]; \\[2ex]
Y_2 = \dfrac{1}{2}\,(M_{31} + M_{24}) = \dfrac{1}{2}\left[\begin{array}{cc|c} 0 & -iI^{(8)} & \\ iI^{(8)} & 0 & 0 \\ \hline & 0 & 0 \end{array}\right]; \\[2ex]
Z_2 = \dfrac{1}{2}\,(M_{31} - M_{24}) = \dfrac{1}{2}\left[\begin{array}{cc|cc} & & & \\ & 0 & & 0 \\ \hline & & 0 & -iI^{(8)} \\ & 0 & iI^{(8)} & 0 \end{array}\right]; \\[2ex]
Y_3 = \dfrac{1}{2}\,(M_{12} + M_{34}) = \dfrac{1}{2}\left[\begin{array}{cc|c} I^{(8)} & 0 & \\ 0 & -I^{(8)} & 0 \\ \hline & 0 & 0 \end{array}\right]; \\[2ex]
Z_3 = \dfrac{1}{2}\,(M_{12} - M_{31}) = \dfrac{1}{2}\left[\begin{array}{cc|cc} & & & \\ & 0 & & 0 \\ \hline & & I^{(8)} & 0 \\ & 0 & 0 & -I^{(8)} \end{array}\right].
\end{array}\right. \tag{18}$$

$$(19)\qquad \begin{cases} [Y_i,\ Y_j] = iY_k\,, \\ [Z_i,\ Z_j] = iZ_k\,; \\ [Y_i,\ Z_j] = 0\,. \end{cases} \qquad (i, j, k = \text{cycl. } 1, 2, 3),$$

Let us also introduce the two sets of generators, $\boldsymbol{Y}^{(K)}$ and $\boldsymbol{Z}^{(K)}$, operating on the K-mesons. These are constructed in the Appendix, where a formalism based on an idea of HARISH-CHANDRA [13] has been employed. Writing the K-meson operator as a column vector (cf. (24))

$$\Phi^{(K)} = \begin{bmatrix} K^+ \\ \bar{K}^+ \\ K^0 \\ \bar{K}^0 \end{bmatrix},$$

we have

$$(20)\qquad \begin{cases} Y_1^{(K)} = \frac{1}{2}\left[\begin{array}{cc|cc} & 0 & 0 & -1 \\ & & 1 & 0 \\ \hline 0 & 1 & & \\ -1 & 0 & & 0 \end{array}\right]; & Z_1^{(K)} = \frac{1}{2}\left[\begin{array}{cc|cc} & 0 & -1 & 0 \\ & & 0 & 1 \\ \hline -1 & 0 & & \\ 0 & 1 & & 0 \end{array}\right]; \\ Y_2^{(K)} = \frac{1}{2}\left[\begin{array}{cc|cc} & 0 & 0 & i \\ & & i & 0 \\ \hline 0 & -i & & \\ -i & 0 & & 0 \end{array}\right]; & Z_2^{(K)} = \frac{1}{2}\left[\begin{array}{cc|cc} & 0 & i & 0 \\ & & 0 & i \\ \hline -i & 0 & & \\ 0 & -i & & 0 \end{array}\right]; \\ Y_3^{(K)} = \frac{1}{2}\left[\begin{array}{cc|cc} 1 & 0 & & 0 \\ 0 & -1 & & \\ \hline & 0 & 1 & 0 \\ & & 0 & -1 \end{array}\right]; & Z_3^{(K)} = \frac{1}{2}\left[\begin{array}{cc|cc} 1 & 0 & & 0 \\ 0 & -1 & & \\ \hline & 0 & -1 & 0 \\ & & 0 & 1 \end{array}\right]. \end{cases}$$

For the π these quantities have the value zero, just as the isobaric spin is zero for the K in this theory.

The Y_i generate infinitesimal rotations in a 3-dimensional subspace of the 4-dimensional hypercharge space. This subspace has been named the *hypercharge-spin space* and $\boldsymbol{Y}$ the *hypercharge spin*. We notice that Y_3 is diagonal in our representation both for the baryons as well as for the K-mesons, and its eigenvalues give half the hypercharge U. The Z_i generate infinitesimal rotations in another 3-dimensional subspace called the *hypernumber-spin space*, $\boldsymbol{Z}$ being called the *hypernumber-spin*.

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[13] HARISH-CHANDRA: *Proc. Roy. Soc.*, A **186**, 502 (1946).

Using these generators, it is now easy to show the formal invariance of the interaction (15) under infinitesimal rotations in the above two 3-dimensional subspaces. The method employed is given in the Appendix. This invariance implies that the total hypercharge spin $\boldsymbol{Y}+\boldsymbol{Y}^{(K)}$ and the total hypernumber spin $\boldsymbol{Z}+\boldsymbol{Z}^{(K)}$ commute with the Hamiltonian and hence are constants of the motion. In other words, this expresses the *hypercharge-independence* and the *hypernumber-independence* of the theory.

We have already interpreted the eigenvalues of Y_3 as half the hypercharge. The quantum numbers associated with the hypernumber-spin conservation will be interpreted only at a later stage.

We give in Table I the quantum numbers characterizing the different particles in this scheme.

TABLE I. - In this table we give the various quantum numbers of the particles and we have also given the values of M_{12} and M_{34} which were called J_3 and J_3' by TIOMNO [4], ζ_3 and ω_3 by DALLAPORTA [5] and $S_1+N/2$ and $S_2+N/2$ by PAIS [6] (N being the baryon-number). The *charge* $C=T_3+M_{12}$ and the hypercharge $U=M_{12}+M_{34}$; $M_{12}=Y_3+Z_3$, $M_{34}=-Y_3-Z_3$. *Note that for the pions, the values in the Table refer to* T_3 *and not to* $\frac{1}{2}T_3$.

Particle name	C	U	Y	Z	Y_3	Z_3	M_{12}	M_{34}	$\frac{1}{2}T_3$	$I_3=\frac{1}{2}T_3+Z_3$	I
p	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
n	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	
Σ^+	1	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1 (Σ)
Y^0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	
Z^0	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0 (Λ)
Σ^-	-1	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	
Ξ^0	0	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Ξ^-	-1	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	
π^+	1	0	0	0	0	0	0	0	1	1	1
π^0	0	0	0	0	0	0	0	0	0	0	
π^-	-1	0	0	0	0	0	0	0	-1	-1	
K^+	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
K^0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	
$\bar{K}^0$	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	-1	0	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{K}^+\equiv K^-$	-1	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0	0	$-\frac{1}{2}$	

We might make here some interesting observations on the physical meaning of these quantum numbers. The emission or absorption of any K-particle by a baryon is accompanied by a simultaneous change of Y, Y_3 and Z, Z_3 for the baryon. A transition in which only Y_3 changes requires a change by 2

of the hypercharge (or strangeness) of the baryon and therefore connects only the $\mathcal{N}$ and the Ξ states via the emission or absorption of a pair of K's or a pair of $\overline{\mathrm{K}}$'s. Instead, the transitions in which only Z_3 changes connect the hyperons among themselves, with no change of the strangeness, through the emission or absorption of the pair of a K and a $\overline{\mathrm{K}}$. This points to a certain physical analogy between the transition in which only Z_3 changes and the π-transitions.

2. – The Ω_K matrices introduced in Section 1 have the same form and commutation relations as the usual Dirac matrices. In the X-representation, Ω_5 has to be diagonal in order that the K-interactions only connect the group of $\mathcal{N}\Xi$-states with the hyperon group and not operate within these groups; that is, in order that the four Ω_K are non-diagonal. In A, the question was raised whether we can choose in this representation another set of such suitable matrices, Ω'_K. This was found to be possible with the following choice:

$$
\left|\begin{aligned}
&\Omega'_1 = -i\Omega_2\Omega_5\,,\\
&\Omega'_2 = \;\; i\Omega_1\Omega_5\,,\\
&\Omega'_3 = -i\Omega_4\Omega_5\,,\\
&\Omega'_4 = \;\; i\Omega_3\Omega_5\,;\\
&\Omega'_5 = \;\; \Omega_5\,;\\
&\Omega'_K\Omega'_L + \Omega'_L\Omega'_K = 2\delta_{KL}I^{(32)}\,,\\
&\Omega_5\Omega'_K + \Omega'_K\Omega_5 = 0\,,\\
&G_\mu\Omega'_K - \Omega'_K G_\mu = 0\,, \qquad (K=1,3)\\
&G_\mu\Omega'_K + \Omega'_K G_\mu = 0\,, \qquad (K=2,4)\\
&G_5\Omega'_K - \Omega'_K G_5 = 0\,.
\end{aligned}\right. \tag{21}
$$

We could have used these Ω'_K instead of the Ω_K and obtained as before a K-interaction scheme quite similar to the one already obtained in Section 1. Let us consider, however, a K-interaction obtained by linearly combining these two types of terms, one with the Ω_K and the other with the Ω'_K, both coupled to the *same* φ_K. It is interesting to note that when the $\overline{X}\Omega_K X$ transform under rotations as the components of a vector in the 4-dimensional hypercharge space, the $\overline{X}\Omega'_K X$ do not so transform since we *insist* on the defining relations (21) in all the rotated frames. Hence the quantity

$$\sum_{L=1}^{4}\varphi_L(\overline{X}\Omega_L X + \overline{X}\Omega'_L X)$$

cannot be invariant under all rotations of the 4-dimensional hypercharge space. Such an interaction could thus be non-hypercharge-independent and hence could possibly lead to the splitting of the $\mathcal{N}$ and the Ξ masses.

This idea has, in fact, been employed in A and B, to explain the $\mathcal{N} - \Xi$ mass-difference. The interaction Lagrangian density is then taken in the X-representation, to be

$$\mathscr{L}_{\text{int}} = \frac{ie}{2}\,\overline{X}(G'_\mu + G''_\mu)A_\mu X + \sum_{K=1}^{4}\overline{X}(iF\Omega_\mu + iF'\Omega'_\mu G_5)G_5^{K+1}X\varphi_K + \tag{22}$$

$$+ ig\sum_{j=1}^{3}\overline{X}T_jG_5^{j+}\,X\Pi_j\,,$$

where F' is an independent real coupling constant.

To see the meaning of these interactions we go to the completely separated representation ψ (cf. equations (9), (10)). We get

$$\begin{aligned}
\mathscr{L}_{\text{int}} = {}& \overline{\psi}_{\text{p}}\{+ie\gamma_\mu A_\mu\psi_{\text{p}} + ig(\sqrt{2}\pi^+\gamma_5\psi_{\text{n}} + \pi^0\gamma_5\psi_{\text{p}}) - \\
& - \sqrt{2}(F - F')(K^+\gamma_5\psi_{\Sigma^0} + K^0\gamma_5\psi_{\Sigma^+})\} + \\
& + \overline{\psi}_{\text{n}}\{+ig(\sqrt{2}\pi^-\gamma_5\psi_{\text{p}} - \pi^0\gamma_5\psi_{\text{n}}) - \sqrt{2}(F - F')(K^+\gamma_5\psi_{\Sigma^-} + K^0\gamma_5\psi_{\Sigma^0})\} + \\
& + \overline{\psi}_{\Xi^0}\{+ig(\sqrt{2}\pi^+\gamma_5\psi_{\Xi^-} + \pi^0\gamma_5\psi_{\Xi^0}) - \sqrt{2}(F + F')(\overline{K}^+\gamma_5\psi_{\Sigma^+} - \overline{K}^0\gamma_5\psi_{\Sigma^0})\} + \\
& + \overline{\psi}_{\Xi^-}\{-ie\gamma_\mu A_\mu\psi_{\Xi^-} + ig(\sqrt{2}\pi^-\gamma_5\psi_{\Xi^0} - \pi^0\gamma_5\psi_{\Xi^-}) - \\
& - \sqrt{2}(F + F')(\overline{K}^+\gamma_5\psi_{\Sigma^0} - \overline{K}^0\gamma_5\psi_{\Sigma^-})\} + \\
& + \overline{\psi}_{\Sigma^+}\{+ie\gamma_\mu A_\mu\psi_{\Sigma^+} + ig(\sqrt{2}\pi^+\gamma_5\psi_{\Sigma^0} + \pi^0\gamma_5\psi_{\Sigma^+}) + \\
& + \sqrt{2}(F + F')K^+\gamma_5\psi_{\Xi^0} + \sqrt{2}(F - F')\overline{K}^0\gamma_5\psi_{\text{p}}\} + \\
& + \overline{\psi}_{\Sigma^0}\{+ig(\sqrt{2}\pi^-\gamma_5\psi_{\Sigma^+} - \pi^0\gamma_5\psi_{\Sigma^0}) + \\
& + \sqrt{2}(F + F')K^+\gamma_5\psi_{\Xi^-} + \sqrt{2}(F - F')\overline{K}_0\gamma_5\psi_{\text{n}}\} + \\
& + \overline{\psi}_{\Sigma^0}\{+ig(\sqrt{2}\pi^+\gamma_5\psi_{\Sigma^-} + \pi^0\gamma_5\psi_{\Sigma^0}) + \\
& + \sqrt{2}(F - F')\overline{K}^+\gamma_5\psi_{\text{p}} - \sqrt{2}(F + F')K^0\gamma_5\psi_{\Xi^0}\} + \\
& + \overline{\psi}_{\Sigma^-}\{-ie\gamma_\mu A_\mu\psi_{\Sigma^-} + ig(\sqrt{2}\pi^-\gamma_5\psi_{\Sigma^0} - \pi^0\gamma_5\psi_{\Sigma^-}) + \\
& + \sqrt{2}(F - F')\overline{K}^+\gamma_5\psi_{\text{n}} - \sqrt{2}(F + F')K^0\gamma_5\psi_{\Xi^-}\}\,,
\end{aligned} \tag{23}$$

$$(\overline{\psi}_{\text{p}} = \psi^\dagger_{\text{p}}\hat{\gamma}_4\,;\ \ldots)\,,$$

where we have, using the particle symbols for the meson operators,

$$
(24)\quad
\begin{cases}
K^+ = \dfrac{\varphi_1 - i\varphi_2}{\sqrt{2}} & \text{(operator for the absorption of } K^+ \text{ and emission of } K^-)\,, \\
\bar{K}^+ = \dfrac{\varphi_1 + i\varphi_2}{\sqrt{2}} & \text{(operator for the absorption of } K^- \text{ and emission of } K^+)\,, \\
K^0 = \dfrac{\varphi_3 - i\varphi_4}{\sqrt{2}} & \text{(operator for the absorption of } K^0 \text{ and emission of } \bar{K}^0)\,, \\
\bar{K}^0 = \dfrac{\varphi_3 + i\varphi_4}{\sqrt{2}} & \text{(operator for the absorption of } \bar{K}^0 \text{ and emission of } K^0)\,, \\
\pi^+ = \dfrac{\Pi_1 - i\Pi_2}{\sqrt{2}} & \text{(operator for the absorption of } \pi^+ \text{ and emission of } \pi^-)\,, \\
\pi^- = \dfrac{\Pi_1 + i\Pi_2}{\sqrt{2}} & \text{(operator for the absorption of } \pi^- \text{ and emission of } \pi^+)\,, \\
\pi^0 = \Pi_3 & \text{(operator for the absorption and emission of } \pi^0)\,.
\end{cases}
$$

We see from (23), that as a result of combining the Ω_K and Ω'_K terms, the coupling constant for the K-interaction turns out to be $F - F'$ for the $\mathcal{N}$ and $F + F'$ for the Ξ. In the K-interactions of the hyperons, instead, both $F + F'$ and $F - F'$ occur as the coupling constants. This, in principle, leads to the separation of the $\mathcal{N}$ and the Ξ masses, leaving the hyperon mass intermediate between them.

The break down of the hypercharge-*independence*, which is responsible for the above mass-splitting, is expressed by the fact that now the K-interaction is no longer invariant under rotations about the axes 1 and 2 of the hypercharge-spin space. It is still invariant under rotations about the *third* axis of this space as well as under any rotation in the hypernumber-spin space. In other word, only $Y_3 + Y_3^{(K)}$, and not the total $\boldsymbol{Y} + \boldsymbol{Y}^{(K)}$ is a constant of the motion besides the total $\boldsymbol{Z} + \boldsymbol{Z}^{(K)}$. The hypernumber-independence is, of course, still to be expected since the new Ω'_K-term introduced above does not suffice to split the hyperon masses.

At the present stage, the introduction of the Ω'_K is really more or less phenomenological and justified only by its consequences. A deeper mathematical explanation for it is still under investigation.

Before closing this section, we should like to make two more observations about the interaction (23). Firstly, we notice that the electromagnetic interaction, as introduced above, represents quite correctly the charges of the various baryon states. Secondly, the pion-interactions are seen to be charge-independent for all the baryon doublets.

3. – We have so far introduced nothing essentially new beyond what has already been done in *A* and *B*. We are still in the doublet approximation as the actual $\Lambda - \Sigma$ mass-difference has not yet been taken into account. We proceed now to introduce this important physical feature into our theory.

In the doublet approximations discussed above, the two fictitious neutral particles Y^0 and Z^0 have quite different physical properties, as each of them is connected by completely different interactions with the other baryons. It is just this physical discrimination between the Y^0 and the Z^0 that is responsible for the already mentioned unacceptable consequences of this approximation in forbidding well observed fast reactions. We try, therefore, to wash out this difference in behaviour of these two particles. We shall do this by considering also a second K- and π-interaction scheme of exactly the same form as before, but in which the roles of the Y^0 and Z^0 are interchanged. This interchange is represented in Fig. 2.

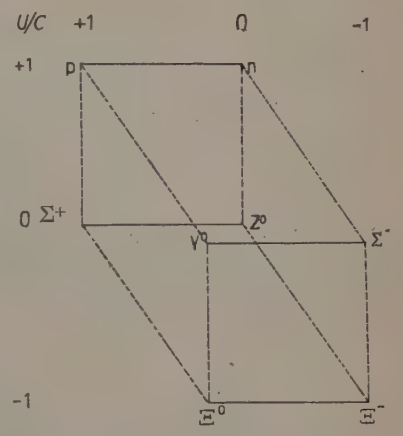

Fig. 2. – The unbroken lines represent the π-transition, the broken lines the transitions involving the neutral K-mesons, and the dotted lines those involving the charged K-mesons. *C* is the charge and *U* the hypercharge of the various baryons.

Mathematically the interchange

$$Y^0 \leftrightarrow Z^0$$

is effected by means of the unitary matrix

$$J = J^\dagger = J^{-1} = \begin{bmatrix} I^{(1)} & 0 & 0 & 0 & 0 \\ 0 & I^{(4)} & & & \\ 0 & & I^{(4)} & 0 & 0 \\ & 0 & 0 & I^{(4)} & \\ 0 & & 0 & & I^{(4)} \end{bmatrix}, \tag{25}$$

operating on $\psi^{(32)}$ in the completely separated representation. The corresponding matrix in the $X^{(32)}$-representation turns out to be again the above *J*, but is different in the intermediate representation $\Psi^{(32)}$. The resulting Lag-

rangian density for the π- and K-interactions in the X-representation is:

$$\bar{\mathscr{L}}_{\text{int}} = \sum_{K=1}^{8} \bar{X}(iF\bar{\Omega}_K + iF'\bar{\Omega}_K\bar{G}_5)\bar{G}_5^{K+1}X\varphi_K + ig\sum_{j=1}^{3}\bar{X}\bar{T}_j\bar{G}_5^{j+1}X\Pi_j\,, \tag{26}$$

where

$$\bar{\Omega}_K = J\Omega_K J\,, \quad \text{etc.} \tag{27}$$

To achieve the desired effect we take the actual interaction Lagrangian of the system to be a linear combination of (22) and (26),

$$L_{\text{int}} = \mathscr{L}_{\text{int}} + b\bar{\mathscr{L}}_{\text{int}}\,, \tag{28}$$

where b is a suitable real constant characteristic for the theory.

To see the effect of this procedure we go again to the completely separated representation ψ, and put

$$\begin{cases} \psi_{\Upsilon^0} = \dfrac{\Sigma^0 + \Lambda^0}{\sqrt{2}}\,, \\ \psi_{Z^0} = \dfrac{\Sigma^0 - \Lambda^0}{\sqrt{2}}\,. \end{cases} \tag{29}$$

Further we put, for convenience of writing,

$$\begin{cases} \psi_{\mathrm{p}} = p\,, & \psi_{\mathrm{n}} = n\,, & \psi_{\Xi^0} = \Xi^0\,, \\ \psi_{\Xi^-} = \Xi^-\,, & \psi_{\Sigma^+} = -\Sigma^+\,, & \psi_{\Sigma^-} = \Sigma^-\,; \end{cases} \tag{30}$$

and change K_0 to $-K_0$ (this is only a trivial redefinition of K_0 in terms of φ_3 and φ_4; then $K^{0\dagger} = -\bar{\mathrm{K}}^0$). We obtain finally the π- and K-interactions in the form

$$\begin{aligned} L_{\text{int}}^{(\pi,K)} = {} & ig(1+b)[(\bar{p}\gamma_5 p - \bar{n}\gamma_5 n)\pi^0 + \sqrt{2}(\bar{p}\gamma_5 n\pi^+ + \bar{n}\gamma_5 p\pi^-)] + \\ & + ig(1+b)[(\bar{\Xi}^0\gamma_5\Xi^0 - \bar{\Xi}^-\gamma_5\Xi^-)\pi^0 + \sqrt{2}(\bar{\Xi}^0\gamma_5\Xi^-\pi^+ + \bar{\Xi}^-\gamma_5\Xi^0\pi^-)] - \\ & - ig(1-b)[(\bar{\Sigma}^0\gamma_5\Lambda^0\pi^0 + \bar{\Sigma}^+\gamma_5\Lambda^0\pi^+ + \bar{\Sigma}^-\gamma_5\Lambda^0\pi^-) + \text{herm. conj.}] + \\ & + ig(1+b)[(\bar{\Sigma}^+\gamma_5\Sigma^+ - \bar{\Sigma}^-\gamma_5\Sigma^-)\pi^0 + \\ & + (\bar{\Sigma}^0\gamma_5\Sigma^- - \bar{\Sigma}^+\gamma_5\Sigma^0)\pi^+ + (\bar{\Sigma}^-\gamma_5\Sigma^0 - \bar{\Sigma}^0\gamma_5\Sigma^+)\pi^-] + \\ & + (F-F')(1-b)[(\bar{p}\gamma_5\Lambda^0 K^+ + \bar{n}\gamma_5\Lambda^0 K^0) + \text{herm. conj.}] - \\ & - (F-F')(1+b)[(\bar{p}\gamma_5\Sigma^0 K^+ - \bar{n}\gamma_5\Sigma^0 K^0 + \sqrt{2}\,\bar{n}\gamma_5\Sigma^- K^+ + \\ & \qquad + \sqrt{2}\,\bar{p}\gamma_5\Sigma^+ K^0) + \text{herm. conj.}] - \\ & - (F+F')(1-b)[(\bar{\Xi}^0\gamma_5\Lambda^0\bar{K}^0 + \bar{\Xi}^-\gamma_5\Lambda^0\bar{K}^+) + \text{herm. conj.}] + \\ & + (F+F')(1+b)[(\bar{\Xi}^0\gamma_5\Sigma^0\bar{K}^0 - \bar{\Xi}^-\gamma_5\Sigma^0\bar{K}^+ + \sqrt{2}\,\bar{\Xi}^-\gamma_5\Sigma^-\bar{K}^0 + \\ & \qquad + \sqrt{2}\,\bar{\Xi}^0\gamma_5\Sigma^+\bar{K}^+) + \text{herm. conj.}]\,. \end{aligned} \tag{31}$$

This is exactly of the same form as the well-known d'Espagnat-Prentki Lagrangian density, but with a very special choice of the eight coupling constants, expressed in terms of only four independent parameters g, F, F' and b. We see that as before all the K-interactions of the $\mathcal{N}$ are proportional to $(F - F')$, and those of the Ξ to $(F + F')$. The new feature now is that all the Σ-couplings are proportional to $(1+b)$ whereas all the Λ-couplings to $(1 - b)$. This then leads to the Λ-Σ mass splitting as was desired. This difference in the Λ and the Σ coupling constants is also exactly what we require to remove the forbiddenness of the already mentioned observed reactions.

Our interaction scheme thus possesses, at least qualitatively, all the desired features right from the start.

Let us consider in a somewhat greater detail the properties of the above interaction. We shall compare the quantities $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{T}$ of Sections 1, 2 relevant to the Lagrangian $\mathscr{L}$ with the corresponding quantities $\overline{\boldsymbol{Y}}$, $\overline{\boldsymbol{Z}}$ and $\overline{\boldsymbol{T}}$ relevant to $\overline{\mathscr{L}}$, where

$$\overline{Y}_i = JY_iJ\,, \quad \text{etc.}$$

To do this we make use of the *completely separated* representation, since in it the effect of J in simply interchangiug the roles of the Υ^0 and the Z^0 is the most obvious. We find that

$$\overline{\boldsymbol{Y}} = \boldsymbol{Y}\,, \tag{32}$$

$$\begin{cases} \overline{\boldsymbol{Z}} \neq \boldsymbol{Z}\,, \\ \overline{\boldsymbol{T}} \neq \boldsymbol{T}\,. \end{cases} \tag{33}$$

To illustrate this we give in Table II the eigenvalues of Z_3, $\overline{Z}_3$, T_3 and $\overline{T}_3$ for states Υ^0 and Z^0 (the other baryon states have not been affected). This com-

TABLE II. - Quantum numbers showing the effect of interchanging the roles of the Υ^0 and the Z^0.

Particle	Y	Z	Z_3	$\frac{1}{2}T_3$	$\overline{Z}_3$	$\frac{1}{2}\overline{T}_3$	$I_3=Z_3+\frac{1}{2}T_3=\overline{Z}_3+\frac{1}{2}\overline{T}_3$
Υ^0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
Z^0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0

parison is only a heuristic way of expressing the mathematical situation that our final interaction (28) is now neither invariant under rotations in the isobaric spin space nor under rotations in the hypernumber-spin space. On the other hand we do find (see the Appendix for the method of showing this) that it is invariant under rotations in a 3-dimensional « *effective* »-*isobaric-spin* space in which the generators of infinitesimal rotations are given by

$$I_i = Z_i + \tfrac{1}{2}T_i\,, \qquad (i = 1, 2, 3),$$

which we call the *effective isobaric spin* operators. It is clear that $I_i^{(K)}$ is equal to $Z_i^{(K)}$ and $I_i^{(\pi)}$ is equal to $T_i^{(\pi)}$, since the K have isobaric-spin zero and the π have the hypernumber-spin zero.

It is thus clear that it is now the *total effective-isobaric-spin* which is a constant of the motion. The eigenvalues of I and I_3 have already been given in the last two columns of Table I in anticipation of the present result. We notice that these values coincide with the isobaric spin assignment of the Gell-Mann–Nishijima scheme, where the Λ^0 is a singlet and the Σ's form a triplet. We might again emphasize that, just as in Section **2**, Y_3 (or the hypercharge) is, of course, conserved but there is no hypercharge-independence.

It is interesting to note that the effective-isobaric-spin of the π and the K have quite different origins. The former is entirely the isobaric spin of the scheme in the doublet approximation, whereas the latter originates from what we have called the hypernumber spin and is intimately connected with the 4-dimensional hypercharge space.

In the frame of the doublet approximations of Section **1**, **2**, the K- and the π-interactions are quite independent owing to the commutation of the Ω_K with the T_j. One can then interpret the 4-dimensional hypercharge space which governs the K-interactions as completely independent of the 3-dimensional isospin space governing the π-interactions. This, in fact, was the point of view of TIOMNO and of the paper *A*, where these two spaces were considered as together forming a 7-dimensional space. However, one can also consider the 3-dimensional isospin space as a subspace of the 4-dimensional hypercharge space and identify the former with the hypernumber-spin space. We shall adopt this point of view as offering a clearer understanding of the developments of this paper. We have already pointed out that in the doublet approximation, the transitions which affect only Z_3 (the pure hypernumber transitions) are accomplished by a pair of a K and an anti-K ($\overline{\mathrm{K}}$) without changing the hypercharge of the baryon, and hence, are in some sense analogous to the π-transitions. The analogy is not exact, as is clear from a look at Figs. 1 and 2, so long as we discriminate between the Y^0 and the Z^0. But as soon as we wash out the difference between these two fictitious particles, as done in this section, we find that physically these transitions become of the same type. They are then characterized neither by $\boldsymbol{Z}$ nor by $\boldsymbol{T}$ but by the vector sum $\boldsymbol{Z}+\frac{1}{2}\boldsymbol{T}$, which we have called the *effective-isobaric-spin* $\boldsymbol{I}$. The isobaric spin space and the hypernumber-spin spaces must then be looked upon as really one and the same space.

The main merit of our scheme, to restate the case, is that in it the baryon mass-differences are taken into account without completely breaking all the charge-state symmetries and yet no known fast reactions are forbidden.

4. – We shall devote this section to stating the different symmetries which underlie our scheme of strong interactions as developed above.

1) The dominating feature of the scheme is its charge independence with the various effective isobaric spins coinciding with the assignments made by GELL-MANN and NISHIJIMA. Although no definite proof is available, there is up to now nothing against the validity of charge independence in the strange particle phenomena.

2) Hypercharge (or strangeness) enters the scheme quite naturally and is a constant of the motion, as it should be. The stronger symmetry of hypercharge *independence* is, of course, no longer present as we have broken it to introduce the $\mathcal{N}$-Ξ mass-splitting.

3) It was pointed out in *A* that the doublet approximation Lagrangian is invariant not only under the $\mathscr{C}$ conjugation but also separately under the boson-conjugation $\mathscr{B}$, and the spinor-conjugation $\mathscr{S}$ (see also B.D.F. ([12])) even when the $\mathcal{N}$-Ξ mass-splitting has been introduced, provided adequate transformation properties are attributed to the different coupling constants. Under the same conditions, our final Lagrangian (cf. Section 3) still remains invariant under the separate $\mathscr{B}$-, $\mathscr{S}$-, $\mathscr{C}$-conjugations (*). This is immediately seen from the fact that the operators $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{C}$ given below commute with J (cf. equation (25)). The definitions of these conjugations were given in *A* for the X-representation as:

$$X^{\mathscr{B}} = \mathscr{B}X\,, \qquad X^{\mathscr{S}} = \mathscr{S}\bar{X}^{T}\,, \qquad X^{\mathscr{C}} = \mathscr{C}\bar{X}^{T}\,; \tag{34}$$

$$\left\{\begin{aligned}
\mathscr{B} &= \left[\begin{array}{cc|cc} 0 & u & & \\ u & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & u \\ \multicolumn{2}{c|}{0} & u & 0 \end{array}\right], \\
\mathscr{S} &= \left[\begin{array}{cc|cc} 0 & v & & \\ v & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & v \\ \multicolumn{2}{c|}{0} & v & 0 \end{array}\right], \\
\mathscr{C} &= \left[\begin{array}{cc|cc} w & 0 & & \\ 0 & w & \multicolumn{2}{c}{0} \\ \hline & & w & 0 \\ \multicolumn{2}{c|}{0} & 0 & w \end{array}\right];
\end{aligned}\right. \tag{35}$$

(*) Professor PAIS has pointed out to us that no selection rules could follow from the $\mathscr{B}$ aud $\mathscr{S}$ invariances, as the sign changes in these operations in the case of the φ_K field cannot be attributed to the field itself but to the two constants F and F', which behave differently from each other: we wish to thank him here for his remark.

where

$$(36)\qquad \begin{cases} u = \begin{bmatrix} 0 & I^{(4)} \\ I^{(4)} & 0 \end{bmatrix}, \\ v = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}, \\ w = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}; \end{cases}$$

$I^{(4)}$ being the 4×4 unit matrix and c the usual 4×4 charge conjugation matrix satisfying the relations

$$(37)\qquad \begin{cases} c\gamma_\mu^T = -\gamma_\mu c, \\ c\gamma_5^T = \gamma_5 c. \end{cases}$$

The behaviour under these operations of various quantities occurring in the Lagrangian are given in Table III.

TABLE III. – *The various quantities shown in the first column transform under $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{C}$ to the quantities given in the second, third and the fourth columns, respectively.*

	$\mathscr{B}$	$\mathscr{S}$	$\mathscr{C}$
e	$-e$	e	$-e$
F	F	$-F$	$-F$
F'	$-F'$	F'	$-F'$
g	g	g	g
$\varphi_1+i\varphi_1$	$\varphi_1-i\varphi_2$	$\varphi_1+i\varphi_2$	$\varphi_1-i\varphi_2$
$\varphi_1-i\varphi_2$	$\varphi_1+i\varphi_2$	$\varphi_1-i\varphi_2$	$\varphi_1+i\varphi_2$
$\varphi_3+i\varphi_4$	$-(\varphi_3-i\varphi_4)$	$-(\varphi_3+i\varphi_4)$	$\varphi_3-i\varphi_4$
$\varphi_3-i\varphi_4$	$-(\varphi_3+i\varphi_4)$	$-(\varphi_3-i\varphi_4)$	$\varphi_3+i\varphi_4$
$\Pi_1+i\Pi_2$	$\Pi_1-i\Pi_2$	$\Pi_1+i\Pi_2$	$\Pi_1-i\Pi_2$
$\Pi_1-i\Pi_2$	$\Pi_1+i\Pi_2$	$\Pi_1-i\Pi_2$	$\Pi_1+i\Pi_2$
Π_3	$-\Pi_3$	$-\Pi_3$	Π_3

The physical meaning of the boson-, spinor- and charge-conjugation ($\mathscr{B}$, $\mathscr{S}$, $\mathscr{C}$) is most easily understood if we consider the following model for the baryon. A « *bare neutral baryon* » B is clothed with all possible mesons to give the different observed baryons. B carries the baryon number, whereas the mesons carry the charge and the hypercharge. For instance, the proton is $B+K^+$, the neutron is $B+K^0$, the Ξ^- is $B+K^-$, and so on. Boson conjugation ($\mathscr{B}$)

then means the reversal of the charge and the hypercharge of the meson clothing, without changing the bare baryon B. Thus, under it, the proton goes to the Ξ^-, the neutron to Ξ^0 and so on. Spinor conjugation ($\mathscr{S}$), on the other hand transforms the bare baryon into its anti-particle $\bar{B}$, without affecting the meson clothing, that is without changing the charge and the hypercharge. The proton is thus transformed under $\mathscr{S}$ into the $\bar{\Xi}^-$ (the anti Ξ^-, which is positively charged and has hypercharge $+1$ and baryon numbers -1). The product of $\mathscr{B}$ and $\mathscr{S}$ is the ordinary charge conjugation $\mathscr{C}$, which reverses not only the charge and the hypercharge but also the baryon number, transforming, for instance, the proton into the anti-proton. In our theory the constants F, e and F' transform under $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{C}$ like the baryon number, the electric charge and the hypercharge, respectively.

4) Finally we consider the relations between the eight coupling constants of the d'Espagnat-Prentki scheme as they come out in our theory. Let these eight constants be g_1, g_2, g_3, g_4, f_1, f_2, f_3 and f_4 for the interactions $\mathcal{N}\mathcal{N}\pi$, $\Sigma\Sigma\pi$, $\Lambda\Sigma\pi$, $\Xi\Xi\pi$, $\mathcal{N}\Sigma K$, $\mathcal{N}\Lambda K$, $\Xi\Sigma K$ and $\Xi\Lambda K$, respectively. Then in our theory they satisfy the following relations:

$$
\text{(38)} \qquad \begin{cases} g_1 = g_2 = g_4 = \overline{g}\,, \\ \dfrac{g_3}{\overline{g}} = \dfrac{f_2}{f_1} = \dfrac{f_4}{f_3} = \dfrac{b-1}{b+1}\,, \\ \dfrac{f_3}{f_1} = \dfrac{f_4}{f_2} = \dfrac{F'+F}{F'-F}\,. \end{cases}
$$

We see that excepting for the $\Lambda\Sigma\pi$-interaction, all the other pion interactions have the same coupling constant.

A crucial test for our theory should be to compare the simple relations (36) with the experimental data.

5. – In all the preceding considerations, we have assumed that the strong interactions are invariant under parity transformation and charge conjugation separately, as also appears to be the case from the present experimental data.

It is, however, of some interest to investigate if we can establish any connection between the independent parity conservation and charge-conjugation invariance on the one hand and the inherent charge-state symmetries of our theory on the other. This question has been recently discussed for some simple cases of strong interactions by various authors [14]. The main results at which one has arrived up to now may be summarized, speaking *only* of non-derivative Yukawa type of couplings, as follows.

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[14] V. G. Soloviev: *Žurn. Èksp. Theor. Fiz.*, **33**, 537, 796 (1957); S. N. Gupta: *Can. Journ. Phys.*, **35**, 1309 (1957); G. Feinberg: *Phys. Rev.*, **108**, 878 (1957).

As a starting point one assumes that strong interactions are certainly CP-invariant. One then looks if the requirements of the charge-state symmetries compel the theory to be invariant under C and P separately. It is found that for an interaction of a neutral boson (π^0, for example), CP-invariance automatically leads to the separate C and P invariances. For charged pions interacting with the nucleons, this is not the case However, if one further assumes the interaction to be charge symmetric, then one does obtain the separate conservation of C and P This result holds for charged pions interacting with any baryon *doublets*, but nor for the $\Sigma\Sigma\pi$ and the $\Sigma\Lambda\pi$ interactions (where the baryon is either a triplet or a singlet). For any of the K-interactions again, simple charge symmetry and CP-invariance do not lead to the separate conservation of C and P.

In the doublet approximation, since all the baryons are doublets, charge symmetry is enough for the purpose of ensuring C and P invariances ([15]). We know, of course, that the doublet approximation is certainly unacceptable. So we do not consider it further, but look for some other conditions.

Thus, when charge symmetry is not enough (as for the $\Sigma\Sigma\pi$-interaction) one tries, as a next step, to see if the stronger requirement of charge independence leads to the desired result. In fact, under this requirement, the $\Sigma\Sigma\pi$-interaction also becomes C and P conserving. Still this is not enough for the $\Lambda\Sigma\pi$-interaction and the various K-interactions.

We notice that the stronger the charge-state symmetries, the more compelled is the theory to be parity conserving if CP-invariance is assumed. A stronger symmetry than the usual charge independence is present in our scheme. In fact, it allows the different baryons to be grouped into a single fermion described by a 32-component spinor, and their interaction to be written by using 32×32 matrices, which all possess a definite character of transposition:

$$\bar{O} = G_\mu O^+ G_\mu = \pm O \,.$$

For our 32-component spinor we have:

$$(39) \qquad \begin{cases} X^p(x) = G_4 X(-x)\,, \\ X^c(x) = \mathscr{C}\bar{X}^T(x)\,, \\ X^{cp}(x) = G_4\mathscr{C}\bar{X}^T(-x)\,. \end{cases}$$

Using these operations we may obtain the transformation properties of the various terms of our Lagrangian density of Section 3. The results are presented in Table IV.

([15]) G. FEINBERG and F. GURSEY: *Phys. Rev.*, **114**, 1153 (1959).

TABLE IV. – *Operators in the first column transform under P, C and CP with the signs shown in the table.*

Operator		P	C	CP
$\bar{X}X$		+	+	+
$\bar{X}G_5X$		−	+	−
$\bar{X}G_\mu X$		− ($\mu = 1, 2, 3$) + ($\mu = 4$)	−	+ ($\mu = 1, 2, 3$) − ($\mu = 4$)
$\bar{X}G'_\mu X$; $\bar{X}G''_\mu X$		− ($\mu = 1, 2, 3$) + ($\mu = 4$)	−	+ ($\mu = 1, 2, 3$) − ($\mu = 4$)
$\bar{X}\Omega_K G_5^{K+1}X$; $\bar{X}\bar{\Omega}_K G_5^{K+1}X$	$K = 1$ 2 3 4	−	− + − +	+ − + −
$\bar{X}\Omega_K G_5^K X$; $\bar{X}\Omega_K G_5^K X$	$K = 1$ 2 3 4	+	− + − +	− + − +
$\bar{X}\Omega'_K G_5^{K+1}X$; $\bar{X}\bar{\Omega}'_K G_5^{K+1}X$	$K = 1$ 2 3 4	−	− + − +	+ − + −
$\bar{X}\Omega'_K G_5^K X$; $\bar{X}\bar{\Omega}'_K G_5^K X$	$K = 1$ 2 3 4	+	− + − +	− + − +
$\bar{X}T_j G_5^{j+1}X$; $\bar{X}\bar{T}_j G_5^{j+1}X$	$j = 1$ 2 3	−	+ − +	− + −
$\bar{X}T_j G_5^j X$; $\bar{X}\bar{T}_j G_5^j X$	$j = 1$ 2 3	+	+ − +	+ − +

Further, under CP the K and the π fields transform as follows:

$$\text{(40)} \qquad \begin{cases} \varphi_L(\boldsymbol{x}) \to n_L^{(K)}\varphi_L(-\boldsymbol{x}), & (L = 1, \ldots, 4), \\ \Pi_j(\boldsymbol{x}) \to n_j^{(\pi)}\Pi_j(-\boldsymbol{x}), & (j = 1, 2, 3), \end{cases}$$

where the n's are phase factors with $|n|^2 = 1$.

Let us now write the interaction Lagrangian density as parity violating. We consider here only one term for simplicity, as the proof is quite similar for the other terms:

$$\mathscr{L}^{(K)} = \bar{X}[f_s \Omega_L G_5^L + f_p \Omega_L G_5^{L+1}] X \varphi_L + \bar{X}[f_s^* \overline{\Omega_L G_5^L} + f_p^* \overline{\Omega_L G_5^{L+1}}] X \varphi_L \,. \tag{41}$$

with f_s, f_p complex.

But:

$$\overline{\Omega_L G_5^L} = \Omega_L G_5^L \,, \qquad \overline{\Omega_L G_5^{L+1}} = -\Omega_L G_5^{L+1} \,. \tag{41a}$$

Therefore:

$$\mathscr{L}^{(K)} = X[F_s \Omega_L + i F_p \Omega_L G_5] G_5^L X \varphi_L \,, \tag{41b}$$

where $F_s = f_s + f_s^*$, $F_p = -i(f_p - f_p^*)$ are real coupling constants. It is therefore Hermiticity for the Lagrangian, together with the inherent symmetries of the scheme expressed by conditions (41*a*) which compels the constants F_s and F_p to be real. The proof now follows along the same line as in the case of a neutral boson interacting with a single baryon. Using Table IV and (40), we find that *CP*-invariance of (41*b*) requires the conditions:

$$\begin{cases} F_s = n_L^{(K)} F_s \,, \\ F_p = -n_L^{(K)} F_p \,. \end{cases} \tag{42}$$

Therefore, *either*

$$F_s = 0 \,,$$

or

$$F_p = 0 \,,$$

which means that the interaction must be parity conserving.

Thus, if our theory is found to be supported by experimental data, we shall be able to conclude that the symmetries of the strong interactions are such that P invariance is not an independent requirement but follows automatically from *CP*-invariance.

* * *

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APPENDIX

I. – To study the behaviour of the Hamiltonian under infinitesimal rotations in the 4-dimensional hypercharge space we give here an extension of a formalism for the 3-dimensional isobaric spin space due to KEMMER, POLKINGHORNE and PURSEY ([16]), which is based on an idea of HARISH-CHANDRA ([13]). For greater clarity, we shall first reproduce the formalism for the 3-dimensional isobaric spin space.

I·1. *3-dimensional isobaric spin space.* – Consider the usual charge independent interaction Hamiltonian density for the pion-nucleon system, ignoring for the present purpose the space-time behaviour. This is enough to illustrate the formalism. We have

$$H = g \sum_{j=1}^{3} \bar{N} \tau_i N \Pi_i \,, \tag{43}$$

where

$$N = \begin{pmatrix} p \\ n \end{pmatrix},$$

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{44}$$

Introduce the column matrix for the π-mesons,

$$\Phi^{(\pi)} = \begin{bmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{bmatrix}, \tag{45}$$

with the help of three row-matrice $I_i^{(\pi)}$ $(i = 1, 2, 3)$ such that

$$I_j^{(\pi)} \Phi^{(\pi)} = \Pi_j \,. \tag{46}$$

Using (24) we have

$$I_1^{(\pi)} = \left(\frac{1}{\sqrt{2}} \; 0 \; \frac{1}{\sqrt{2}} \right), \qquad I_2^{(\pi)} = \left(\frac{i}{\sqrt{2}} \; 0 \; \frac{-i}{\sqrt{2}} \right), \qquad I_3^{(\pi)} = (0\,1\,0) \,. \tag{47}$$

Also define the hermitian adjoint (column vectors):

$$T_j^{(\pi)} \equiv I_j^{(\pi)\dagger}, \qquad (j = 1, 2, 3) \tag{48}$$

We have then

$$I_i^{(\pi)} T_j^{(\pi)} = \delta_{ij} \,. \tag{49}$$

([16]) N. KEMMER, J. C. POLKINGHORNE and D. L. PURSEY: *Rep. Prog. Phys.*, **22**, 368 (1959).

We construct now three (3×3)-matrices:

$$\left\{\begin{array}{l} T_1^{(\pi)} = -i(T_2^{(\pi)} \Gamma_3^{(\pi)} - T_3^{(\pi)} \Gamma_2^{(\pi)}), \\ T_2^{(\pi)} = -i(T_3^{(\pi)} \Gamma_1^{(\pi)} - T_1^{(\pi)} \Gamma_3^{(\pi)}), \\ T_3^{(\pi)} = -i(T_1^{(\pi)} \Gamma_2^{(\pi)} - T_2^{(\pi)} \Gamma_1^{(\pi)}). \end{array}\right. \tag{50}$$

Using (49) we see easily that

$$T_i^{(\pi)} T_j^{(\pi)} - T_j^{(\pi)} T_i^{(\pi)} = i T_k^{(\pi)}, \qquad (i, j, k = \text{cycl. } 1, 2, 3), \tag{51}$$

and

$$\Gamma_i^{(\pi)} T_j^{(\pi)} = i \Gamma_k^{(\pi)}, \qquad (i, j, k = \text{cycl. } 1, 2, 3). \tag{52}$$

The $T_i^{(\pi)}$ are, in fact, the isobaric spin matrices for the pion. Written explicitly,

$$\left\{\begin{array}{l} T_1^{(\pi)} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \\ T_3^{(\pi)} = \begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{bmatrix}, \\ T_3^{(\pi)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \end{array}\right. \tag{53}$$

To prove algebraically the invariance of (43) under rotations in the isobaric spin space, consider an *infinitesimal* rotation ε about the jth-axis of this space. Under this rotation N, $\bar{N}$ and $\Phi^{(\pi)}$ transform to

$$\left\{\begin{array}{l} N' = (1 + \frac{1}{2} i\varepsilon\tau_j) N, \\ \bar{N}' = \bar{N}(1 - \frac{1}{2} i\varepsilon\tau_j), \\ \Phi^{(\pi)\prime} = (1 + i\varepsilon T_j^{(\pi)}) \Phi^{(\pi)}; \end{array}\right. \tag{54}$$

and hence H transforms to

$$H' = H + gi\varepsilon \sum_{i=1}^{3} [\bar{N} \tfrac{1}{2} (\tau_i \tau_j - \tau_j \tau_i) N \Gamma_i^{(\pi)} \Phi^{(\pi)} + \bar{N} \tau_i N \Gamma_i^{(\pi)} T_j^{(\pi)} \Phi^{(\pi)}], \tag{55}$$

which using the commutation relations for the τ_i and the relations (52) gives

$$H' = H. \tag{56}$$

This proves formally the invariance of H and hence implies the conservation of the *total* isobaric spin of the system.

I·2. *4-dimensional hypercharge space.* – Let us now study the invariance properties of the interaction (15) of Section 2 under rotation in this space,

$$\mathscr{L}_K = iF \sum_{K=1}^{4} \overline{\Psi}\, \omega_K \Psi \varphi_K \,. \tag{57}$$

We have already constructed in Section 2 the *two* sets of generators, Y_i and Z_i $(i=1, 2, 3)$ of infinitesimal rotations in the two 3-dimensional subspaces (the hypercharge-spin space and the hypernumber-spin space) of this 4-space. These operate on the Ψ. For the K-field we introduce now the column matrix

$$\Phi^{(K)} = \begin{bmatrix} K^+ \\ \overline{K}^+ \\ K^0 \\ \overline{K}^0 \end{bmatrix}, \tag{58}$$

with the help of four row matrices $\Gamma_L^{(K)}$ $(L=1, 2, 3, 4)$ such that

$$\Gamma_L^{(K)} \Phi^{(K)} = \Phi_L \,. \tag{59}$$

From (24) we see that

$$\begin{cases} \Gamma_1^{(K)} = \left(\dfrac{1}{\sqrt{2}}\ \dfrac{1}{\sqrt{2}}\ 0\ 0\right), & \Gamma_2^{(K)} = \left(\dfrac{i}{\sqrt{2}}\ \dfrac{-i}{\sqrt{2}}\ 0\ 0\right), \\ \Gamma_3^{(K)} = \left(0\ 0\ \dfrac{1}{\sqrt{2}}\ \dfrac{1}{\sqrt{2}}\right), & \Gamma_4^{(K)} = \left(0\ 0\ \dfrac{i}{\sqrt{2}}\ \dfrac{-i}{\sqrt{2}}\right). \end{cases} \tag{60}$$

Introducing the hermitian adjoints of the $\Gamma_L^{(K)}$:

$$T_L^{(K)} \equiv \Gamma_L^{(K)\dagger}, \tag{61}$$

we see that

$$\Gamma_K^{(K)} T_L^{(K)} = \delta_{KL} \,. \tag{62}$$

We construct now the six (4×4)-matrices:

$$\begin{cases} Y_1^{(K)} = -\dfrac{i}{2}\left[(T_2^{(K)}\Gamma_3^{(K)} - T_3^{(K)}\Gamma_2^{(K)}) + (T_1^{(K)}\Gamma_4^{(K)} - T_4^{(K)}\Gamma_1^{(K)})\right], \\ Z_1^{(K)} = -\dfrac{i}{2}\left[(T_2^{(K)}\Gamma_3^{(K)} - T_3^{(K)}\Gamma_2^{(K)}) - (T_1^{(K)}\Gamma_4^{(K)} - T_4^{(K)}\Gamma_1^{(K)})\right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{63}$$

(and the other four obtained by a cyclic permutations of 1, 2, 3).

Using (62) we see that

$$(64)\qquad \begin{cases} Y_i^{(K)} Y_j^{(K)} - Y_j^{(K)} Y_i^{(K)} = i\, Y_k^{(K)}\,, \\ Z_i^{(K)} Z_j^{(K)} - Z_j^{(K)} Z_i^{(K)} = i Z_k^{(K)}\,. \end{cases} \qquad (i, j, k = \text{cycl. } 1, 2, 3)\,.$$

$Y_i^{(K)}$ and $Z_i^{(K)}$ are respectively the hypercharge-spin and the hypernumber-spin matrices for the K given in the text by (20). It is again easy, using (62), to obtain the following relations

$$(65)\qquad \begin{cases} \Gamma_i^{(K)} Y_i^{(K)} = -\Gamma_i^{(K)} Z_i^{(K)} = -\dfrac{i}{2}\Gamma_4^{(K)}\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, 3); \\ \Gamma_4^{(K)} Y_i^{(K)} = -\Gamma_4^{(K)} Z_i^{(K)} = \dfrac{i}{2}\Gamma_i^{(K)}\,; \\ \Gamma_i^{(K)} Y_j^{(K)} = -\Gamma_j^{(K)} Y_i^{(K)} = \dfrac{i}{2}\Gamma_k^{(K)}\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad (i, j, k = \text{cycl. } 1, 2, 3)\,. \\ \Gamma_i^{(K)} Z_j^{(K)} = -\Gamma_j^{(K)} Z_i^{(K)} = \dfrac{i}{2}\Gamma_k^{(K)}\,. \end{cases}$$

We now prove the invariance of (57) under rotations in the 4-dimensional hypercharge space. Since these are compounded out of rotations in the two 3-dimensional spaces called here the hypercharge-spin space and the hypernumber-spin space, we shall consider here the latter. Under an infinitesimal rotation ε about the i-th axis ($i = 1, 2, 3$) in the hypercharge-spin space Ψ, $\overline{\Psi}$ and $\Phi^{(K)}$ transform into

$$(66)\qquad \begin{cases} \Psi' = (1 + i\varepsilon Y_i)\Psi\,, \\ \overline{\Psi}' = \overline{\Psi}(1 - i\varepsilon Y_i)\,, \\ \Phi^{(K)\prime} = (1 + i\varepsilon Y_i^{(K)})\,\Phi^{(K)}\,. \end{cases}$$

Then $\mathscr{L}_K$ transforms into

$$(67)\qquad \mathscr{L}_K' = \mathscr{L}_K + i\varepsilon(iF)\sum_{K=1}^{4}[\overline{\Psi}(\omega_K Y_i - Y_i\omega_K)\Psi\Gamma_K^{(K)}\Phi^{(K)} + \overline{\Psi}\omega_K\Psi\Gamma_K^{(K)} Y_i^{(K)}\Phi^{(K)}]\,.$$

Using the commutation relations of the ω_K (and the expressions of the Y_i in terms of the ω_K, (17), (18)) along with the relations (65) we verify that

$$(68)\qquad \mathscr{L}_K' = \mathscr{L}_K\,,$$

proving thereby the invariance of $\mathscr{L}_K$ under rotations in the hypercharge-spin space. Similarly we may, by replacing above the Y's by the Z's, prove the invariance under rotations in the hypernumber-spin space. These then imply

the conservation of the *total* hypercharge-spin and the total hypernumber-spin of the system described by $\mathscr{L}_K$.

This method applied to the various cases considered in this paper.

II. – To make the identity of our « effective » isobaric spin with the isobaric spin assignments of the Gell-Mann-Nishijima scheme apparent to the eye we make the following unitary transformations (note that we are in the completely separated representation):

$$\psi \to R\psi = \begin{bmatrix} p \\ n \\ \cdots\cdots \\ \Xi^0 \\ \Xi^- \\ \cdots\cdots \\ \Lambda^0 \\ \cdots\cdots \\ -\Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{bmatrix}, \tag{69}$$

$$\Phi^{(K)} \to r\Phi^{(K)} = \begin{bmatrix} K^+ \\ -K^0 \\ \cdots\cdots \\ K^0 \\ \overline{K}^+ \end{bmatrix}, \tag{70}$$

$$R = \left[\begin{array}{cc|cc|ccc} 1 & 0 & & & & & \\ 0 & 1 & & 0 & & & \\ \hline & & 1 & 0 & & & \\ & 0 & 0 & 1 & & 0 & \\ \hline & & & & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ & & & & -1 & 0 & 0 \\ & 0 & & & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ & & & & 0 & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right], \tag{71}$$

$$r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{72}$$

Then the transformed « effective » isobaric spin matrices are

$$
(73)\qquad
\begin{cases}
RI_1R^{-1} = \dfrac{1}{2}\begin{bmatrix}
0 & 1 & & & & & & & \\
1 & 0 & & & & & & & \\
 & & 0 & 1 & & & & 0 & \\
 & & 1 & 0 & & & & & \\
 & & & & 0 & & & & \\
 & & & & & 0 & -\sqrt{2} & 0 & \\
 & 0 & & & & -\sqrt{2} & 0 & \sqrt{2} & \\
 & & & & & 0 & \sqrt{2} & 0 &
\end{bmatrix}, \\
RI_2R^{-1} = \dfrac{1}{2}\begin{bmatrix}
0 & -i & & & & & & & \\
i & 0 & & & & & & & \\
 & & 0 & -i & & & & 0 & \\
 & & i & 0 & & & & & \\
 & & & & 0 & & & & \\
 & & & & & 0 & i\sqrt{2} & 0 & \\
 & 0 & & & & -i\sqrt{2} & 0 & -i\sqrt{2} & \\
 & & & & & 0 & i\sqrt{2} & 0 &
\end{bmatrix}, \\
RI_3R^{-1} = \dfrac{1}{2}\begin{bmatrix}
1 & 0 & & & & & & & \\
0 & -1 & & & & & & & \\
 & & 1 & 0 & & & & 0 & \\
 & & 0 & -1 & & & & & \\
 & & & & 0 & & & & \\
 & & & & & 2 & 0 & 0 & \\
 & 0 & & & & 0 & 0 & 0 & \\
 & & & & & 0 & 0 & -2 &
\end{bmatrix},
\end{cases}
$$

$$
(74)\qquad
\begin{cases}
rI_1^{(K)}r^{-1} = \dfrac{1}{2}\begin{bmatrix}
0 & 1 & & 0 & \\
1 & 0 & & & \\
 & & 0 & 1 & \\
 & 0 & 1 & 0 &
\end{bmatrix}, \\
rI_2^{(K)}r^{-1} = \dfrac{1}{2}\begin{bmatrix}
0 & -i & & 0 & \\
i & 0 & & & \\
 & & 0 & -i & \\
 & 0 & i & 0 &
\end{bmatrix}, \\
rI_3^{(K)}r^{-1} = \dfrac{1}{2}\begin{bmatrix}
1 & 0 & & 0 & \\
0 & -1 & & & \\
 & & 1 & 0 & \\
 & 0 & 0 & -1 &
\end{bmatrix}.
\end{cases}
$$

(73) are the « *direct sums* » of the usual isobaric spin matrices for the doublet $\mathcal{N}$, the doublet Ξ, the singlet Λ^0 and the triplet Σ. (74) are similarly the direct sums of the isobaric spin matrices for the two K doublets.

RIASSUNTO

Il presente lavoro dà uno schema delle interazioni forti dei barioni, che fin dal principio tiene conto delle differenti masse dei barioni ed ha inoltre un alto grado di simmetria senza proibire alcuna delle reazioni veloci conosciute. Si fa uso soltanto del solito tipo di interazioni fermione-fermione-bosone. Si parte dai risultati già ottenuti nei lavori A e B (9), dove si era fatto uso della approssimazione di doppietto in cui i barioni sono raggruppati in quattro doppietti isobarici aventi la stessa massa. I quattro doppietti sono descritti con un singolo spinore a 32 componenti e le loro interazioni sono descritte usando delle matrici 32×32 che operano su questo spinore. In A la interazione K, tralasciando le differenze di massa, fu presa come invariante rispetto alle rotazioni in uno spazio euclideo quadridimensionale, detto spazio di ipercarica, dove i mesoni K descritti con quattro campi reali, sono considerati come le componenti di un vettore di questo spazio. L'interazione π era naturalmente invariante rispetto alle rotazioni nell'usuale spazio isobarico tridimensionale. Significativi per questo lavoro sono i sei generatori delle rotazioni infinitesime nello spazio di ipercarica raggruppati qui in due trivettori appaiati Y_i — operatori di ipercarica — e Z_i — operatori di ipernumero — $(i = 1, 2, 3)$, che generano rotazioni infinitesime in due differenti sottospazi tridimensionali dello spazio di ipercarica e sono chiamati rispettivamente « spin di ipercarica e spin di ipernumero ». In A la differenza di massa fra $\mathcal{N}$ e Ξ era introdotta aggiungendo all'interazione K un altro termine, tale che il Lagrangiano risultante non fosse più invariante rispetto alle rotazioni nello spazio di ipernumero. Nel presente lavoro viene compiuto l'ultimo passo per ottenere anche la separazione della massa della Λ e della Σ. Questo è fatto prendendo per il Lagrangiano finale d'interazione π e K, una combinazione lineare del Lagrangiano originale di A, e di uno ottenuto da questo intercambiando le particelle fittizie Y^0 e Z^0 ($Y^0 = (\Sigma^0 + \Lambda^0)/\sqrt{2}$, $Z^0 = (\Sigma^0 - \Lambda^0)/\sqrt{2}$). Con ciò soltanto quattro costanti indipendenti (F, F^1, g e b), sono introdotte nella teoria. Ne risulta che i vari termini del Lagrangiano finale hanno esattamente la forma dei termini del Lagrangiano di d'Espagnat-Prentki. La differenza è che ora le otto costanti di accoppiamento non sono indipendenti, ma sono espresse in termini dei soli quattro parametri F, F^1, g e b. Il presente Lagrangiano non conserva ora nè lo spin di ipernumero totale, nè lo spin isobarico totale dell'approssimativo doppietto, ma conserva la quantità già ottenuta come somma vettoriale dei due, che viene chiamata « spin isobarico effettivo »: ciò ha esattamente lo stesso valore per le diverse particelle, di quelle dateci nello schema di Gell-Mann e Nishijima. Questo ci porta ad identificare lo spazio di spin isobarico con lo spazio di ipernumero e considerarlo come un sottospazio dello spazio di ipercarica quadridimensionale. Troviamo inoltre che la nostra teoria è separatamente invariante per le operazioni di coniugazione bosonica, spinoriale e di carica. Infine viene dimostrato che la simmetria degli stati di carica inerenti alla nostra teoria, sono tali che, assumendo la invarianza CP, se ne deduce automaticamente la conservazione della parità.

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A Theorem on the Elimination of Contact Muon-Electron Interactions.

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Summary. — A general theorem on the elimination of possible contact muon-electron interactions is given which includes as particular cases a theorem by Cabibbo and Gatto and a theorem by Feinberg, Kabir and Weinberg for particular types of interactions.

1. – Introduction.

Gell-Mann and Feynman have remarked (1) that if one considers the expansion of a hypothetical muon-electron interaction in powers of momentum transfer, the decay rate inferred from the leading term of such an expansion by invoking gauge invariance is identically zero. Later, this result was shown (2) by two of us (N.C. and R.G.) to be contained in a general equivalence theorem. The theorem states that weak interactions of the form

$$\bar{\mu}(x)\gamma(\partial - ieA)(1+\gamma_5)e(x) + \text{h. c.}\,, \tag{1}$$

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(1) M. Gell-Mann and R. P. Feynman: *Annual International Conference on High Energy Physics at CERN*, edited by B. Ferretti (Geneva, 1958), p. 261.

(2) N. Cabibbo and R. Gatto: *Phys. Rev.*, **116**, 1134 (1959); see also R. Gatto: *Lectures at the International School of Physics in Varenna*, June 1959 (to appear in *Suppl. Nuovo Cimento*).

where $\mu(x)$ and $e(x)$ denote the muon and electron fields, respectively, can be removed by a canonical transformation. The argument demonstrated the existence of a unitary matrix which transforms the (eight component) spinor ψ,

$$\psi = \begin{bmatrix} e \\ \mu \end{bmatrix}, \tag{2}$$

in such a manner that the transformed Lagrangian no longer contains an interaction term of the type (1). Recently, FEINBERG, KABIR and WEINBERG [3] have noted the possibility of eliminating, by similar methods, hypothetical interactions of the form

$$-\varrho\bar{e}(x)\mu(x) - \xi\bar{e}(x)\gamma(\partial - ieA)\mu(x) + \text{H. c.} \tag{3}$$

or, alternatively, of the form

$$-\varrho\bar{e}(x)\gamma_5\mu(x) - \xi\bar{e}(x)\gamma\gamma_5(\partial - ieA)\mu(x) + \text{H. c.} \tag{3'}$$

to all orders in the coupling constants ϱ and ξ.

In this note, we present a generalization and unification of such arguments. The results described above appear as particular cases of a principle applicable to the most general renormalizable interactions of muons, electrons and photons. Our principal theorem is stated and proved in the next section. The conclusion does not depend on any perturbation assumption. We do assume that the energy operator deduced from the Lagrangian is positive definite. The argument utilizes a theorem on the diagonalization of finite dimensional matrices whose proof is given in Section 3.

2. – General form of the equivalence theorem.

We consider a Lagrangian of the type

$$\mathscr{L} = -\overline{\psi}[\gamma_\mu(\partial_\mu - ieA_\mu)(A + \gamma_5 B) + C + i\gamma_5 D]\psi - \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \mathscr{L}_s\,, \tag{4}$$

where ψ is defined in (2). The two-by-two matrices A, B, C and D operate in the $\mu - e$ space (which we shall call lepton space or, simply, L-space); $-(1/4)F_{\mu\nu}F_{\mu\nu}$ is the free photon Lagrangian, and $\mathscr{L}_s$ is the Lagrangian for strongly interacting particles. We adopt the customary assumption that $\mathscr{L}_s$ does not contain the electron or muon fields. In order that $\mathscr{L}$ be Hermitian, A, B, C and D must be Hermitian matrices. Then Eq. (4) represents the

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[3] G. FEINBERG, P. KABIR and S. WEINBERG: *Phys. Rev. Lett.*, **3**, 527 (1959).

most general Lagrangian containing renormalizable interactions and neglecting weak interactions.

We now assert that by means of a suitable non-singular matrix transformation in spin space and L-space,

$$\psi = \boldsymbol{T}\psi^r , \tag{5}$$

the Lagrangian (4) may be brought into a form in which the electron and muon components of ψ^r are not coupled, and such that the electron and muon terms in $\mathscr{L}$ are of the canonical type for spin 1/2 fields.

It is convenient to use the projections $a = \frac{1}{2}(1+\gamma_5)$ and $\bar{a} = \frac{1}{2}(1-\gamma_5)$. Then

$$A + B\gamma_5 = (A+B)a + (A-B)\bar{a} , \tag{6}$$

$$C + iD\gamma_5 = (C+iD)a + (C-iD)\bar{a} . \tag{6'}$$

We assume that the energy operator is positive definite. It follows [4] that $A+B\gamma_5$ is positive definite. Since a and $\bar{a}$ are orthogonal, we conclude from (6) that the two dimensional matrices $A+B$ and $A-B$ are positive definite.

From (4), we obtain for the canonical momentum conjugate to ψ,

$$\pi = i\psi^\dagger(A+\gamma_5 B) . \tag{7}$$

One may verify that the canonical anticommutation relations are consistent provided that

$$\det(A+\gamma_5 B) = \det(A-B)\det(A+B) \neq 0 . \tag{8}$$

Eq. (8) may also be regarded as the requirement that ψ does not obey an equation of constraint. Its validity is assured by the positive definite property.

The matrix $\boldsymbol{T}$ of eq. (5) may, if it exists, be written

$$\boldsymbol{T} = aR + \bar{a}S , \tag{9}$$

where R and S act in L-space. We require that in terms of ψ^r and $\bar{\psi}^r$, de-

[4] One may adopt the procedure used, for example, by N. N. BOGOLJUBOV: *Introduction to the Theory of Quantized Fields*, (New York, 1959) p. 123, for free spinor fields. One obtains the energy as an integral over terms of the type $a^\dagger_\pm(\boldsymbol{k})(A+B\gamma_5)\cdot a_\pm(\boldsymbol{k})|k_0|$ where the $a_\pm(\boldsymbol{k})$ are essentially Fourier components of the operators $\psi^\pm(x)$. The relationship between the positive definiteness of the energy and of $(A+B\gamma_5)$ then follows directly.

fined by

$$\psi = (aR + \bar{a}S)\psi' , \tag{10}$$

$$\bar{\psi} = \bar{\psi}'(\bar{a}R^\dagger + aS^\dagger) , \tag{10'}$$

the Lagrangian (4) assume the form

$$\mathscr{L} = -\bar{\psi}'[\gamma_\mu(\partial_\mu - ieA_\mu) + M]\psi' - \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \mathscr{L}_s , \tag{11}$$

where M is a diagonal matrix in L-space. We demand that $\boldsymbol{T}$ be chosen so that the diagonal elements of M, which represent the electron and muon masses in the new representation, be real and non-negative.

Inserting (10) and (10′) into (4) and comparing with (11), we infer the following conditions on R and S:

$$R^\dagger(A + B)R = 1 , \qquad S^\dagger(A - B)S = 1 , \tag{12}$$

$$S^\dagger(C + iD)R = M , \qquad R^\dagger(C - iD)S = M . \tag{13}$$

The two equations of (13) are Hermitian conjugates of each other; therefore only one of them need be considered.

We first construct R and S so as to satisfy (12). Let V_+ and V_- be the unitary matrices which diagonalize the (Hermitian) matrices $A+B$ and $A-B$. Thus

$$V_+^{-1}(A + B)V_+ = G_+ , \qquad V_-^{-1}(A - B)V_- = G_- , \tag{14}$$

where G_+, G_- are diagonal and positive definite. We also introduce the two matrices

$$T_+ = (G_+)^{-\frac{1}{2}} , \qquad T_- = (G_-)^{-\frac{1}{2}} . \tag{15}$$

We then observe that if U_+ and U_- denote *arbitrary* unitary matrices, the choices

$$R = V_+T_+U_+ , \qquad S = V_-T_-U_- \tag{16}$$

satisfy eq. (12). With these choices, the first equation of (13) becomes

$$U_-^{-1}[T_-V_-^{-1}(C + iD)V_+T_+]U_+ = M . \tag{17}$$

The existence of unitary matrices U_+ and U_- which satisfy (17) is a direct consequence of a theorem we prove in Section 3. Therefore, the existence of

R and S, and hence of $\boldsymbol{T}$ is also proved. Since $V_{\pm}$, $T_{\pm}$, and $U_{\pm}$ are non-singular, R, S and hence $\boldsymbol{T}$ are likewise non-singular and possess inverses. This establishes the theorem.

We now comment on the interpretation of the theory. We suppose that a non-singular operator T corresponds to the matrix $\boldsymbol{T}$ in the sense that

$$T^{-1}\psi' T = \psi = \boldsymbol{T}\psi' . \tag{18}$$

Let P_μ be the four-momentum operator constructed from (4) and let P'_μ be the corresponding operator constructed from (11). Then $P_\mu = T^{-1} P'_\mu T$. One sees that

$$P'_\mu P'_\mu |\alpha\rangle = -\mu^2 |\alpha\rangle \tag{19}$$

has solutions corresponding to the eigenvalues $\mu^2 = m_e^2$, $\mu^2 = m_\mu^2$, where m_e and m_μ are the diagonal elements of the non-negative matrix M. Hence $P_\mu P_\mu$ has eigenstates $T^{-1}|\alpha\rangle$ with the same eigenvalues $-\mu^2$. The theories defined by (4) and (11) are completely equivalent and both describe the same physical situation. The form (11) has the advantage of corresponding to the simplest limiting relation between the interpolating Heisenberg field ψ' and the asymptotic « in » fields $e^{(\mathrm{in})}(x)$ and $\mu^{(\mathrm{in})}(x)$. This remark aids in clarifying the question of the formal interpretation of symmetry operations in the theory (such as parity, time inversion, universality, etc.). For instance, the Lagrangian (4) is invariant under space inversion even if A, C and B, D are simultaneously non zero; it is also invariant under time reversal even if A, B, C, D are not all real, etc.: the point is that such symmetry operations have a direct interpretation on the asymptotic fields and they must be correspondingly redefined when the interpolating Heisenberg fields do not satisfy the proper limiting conditions.

3. – A diagonalization theorem for finite dimensional matrices.

We recall two simple facts about finite dimensional matrices:

(i) If the matrix Q commutes with its adjoint, it is diagonalizable by a unitary transformation;

(ii) If F is an arbitrary matrix, the matrices $FF^\dagger$ and $F^\dagger F$ have the same eigenvalues.

The equation $QQ^\dagger = Q^\dagger Q$ implies that the Hermitian and skew Hermitian parts of Q commute and hence can be diagonalized by the same unitary matrix. Then Q itself is diagonalized by this matrix, confirming (i). To verify (ii), we

note that the eigenvalues are determined by the equations

$$\det(FF^\dagger - \lambda I) = 0 \tag{20}$$

and

$$\det(F^\dagger F - \lambda I) = 0\,. \tag{20'}$$

If $\det F \neq 0$,

$$\det(FF^\dagger - \lambda I) = \det F \det(F^\dagger - \lambda F^{-1}) = \det(F^\dagger F - \lambda I)\,,$$

so that eq. (20) and (20′) are identical. If $\det F = 0$, we define $F(\varepsilon) = F - \varepsilon I$. For all $\varepsilon \neq 0$ in a small interval around $\varepsilon = 0$, $\det F(\varepsilon) \neq 0$ and hence $\det(F(\varepsilon)F^\dagger(\varepsilon) - \lambda I) = \det(F^\dagger(\varepsilon)F(\varepsilon) - \lambda I)$. This is a relation between two polynomials in ε and in the limit $\varepsilon \to 0$, we obtain the desired result.

We now prove the diagonalization theorem.

Theorem: For any matrix F there exist unitary matrices U_1 and U_2 such that

$$U_1^{-1} F U_2 = Z\,,$$

where Z is a non-negative definite diagonal matrix.

Proof: Let W_1 and W_2 be the unitary matrices which diagonalize the Hermitian matrices $FF^\dagger$ and $F^\dagger F$. Since these matrices have the same eigenvalues, one can choose W_1 and W_2 so that

$$W_1^{-1} F F^\dagger W_1 = W_2^{-1} F^\dagger F W_2\,. \tag{21}$$

Let us define $Q = W_1^{-1} F W_2$. It follows that $Q^\dagger = W_2^{-1} F^\dagger W_1$ and (21) can be written

$$QQ^\dagger = Q^\dagger Q\,. \tag{22}$$

Therefore, by (i), there exists a unitary matrix V such that

$$V^{-1} Q V = V^{-1} W_1^{-1} F W_2 V = (W_1 V)^{-1} F (W_2 V) = Z'\,, \tag{23}$$

where Z' is diagonal. One can always write Z' in the form

$$Z' = Z V'\,, \tag{24}$$

where Z is the diagonal non-negative definite matrix obtained by replacing the elements of Z' by their absolute values, and V' is diagonal and unitary.

If now one takes

$$U_1 = W_1 V\,,$$

$$U_2 = W_2 V (V')^{-1}$$

one finds

$$U_1^{-1} F U_2 = (W_1 V)^{-1} F (W_2 V)(V')^{-1} = Z'(V')^{-1} = Z$$

as desired.

RIASSUNTO

Viene dimostrato un teorema generale che permette l'eliminazione di eventuali interazioni locali μ-e e che contiene come casi particolari un teorema di Cabibbo e Gatto ed un teorema di Feinberg, Kabir e Weinberg.

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β-γ Directional Correlation in ^{152}Eu.

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Summary. — The directional correlation between the first forbidden 1 483 keV β-group and the 344 keV $E2$ γ-transition in the decay of 13-year ^{152}Eu has been measured as a function of β-energy above 950 keV. The integral β-γ correlation data for β energies above 950 keV can be fitted with a directional correlation function

$$W(\theta) = 1 - (0.379 \pm 0.004) \cos^2 \theta \, .$$

The anisotropy is negative and is found to increase with β-energy, its value is -0.416 ± 0.012 at 1 350 keV, the maximum energy at with measurements were made. The energy dependence of the anisotropy excludes the unique character of the β-transition. The observed energy dependence of the anisotropy is fitted with the $(V-A)$ theory in the modified B_{ij} approximation yielding two values of the matrix element ratio $Y = -(C_V/C_A)(\int i\boldsymbol{\alpha}/\int B_{ij}) \approx 1.05$ or 0.09. The former value of Y, however, reproduces closely the spectral shape recently observed for the 1 483 keV β-group.

1. – Introduction.

Earlier measurements of the shape of the highest energy β-group of 1 483 keV in the 13-year ^{152}Eu have suggested that the spectrum has an « unique » shape ([1,2]) or an « allowed » shape ([3]). Recently Langer *et al* ([4]),

([1]) J. M. Cork, M. K. Brice, R. G. Helmer and D. E. Sarason: *Phys. Rev.*, **107**, 1621 (1957).

([2]) S. K. Bhattacherjee, T. D. Nainan, S. Raman and B. Sahai: *Nuovo Cimento*, **7**, 501 (1958).

([3]) D. E. Alburger, S. Ofer and M. Goldhaber: *Phys. Rev.*, **112**, 1998 (1958).

([4]) L. M. Langer, D. R. Smith and M. P. Klein: *Bull. Am. Phys. Soc.*, **4**, 426 (1959).

using a high resolution spectrometer and employing a relatively thin source, have found that the shape is neither « allowed » nor « unique ». The spin of the ground state of ^{152}Eu has been measured ([5]) to be $J = 3$. The log (ft) value for this transition is about 12 which is abnormally high for a first-forbidden transition of $\Delta J = 1$, Yes type.

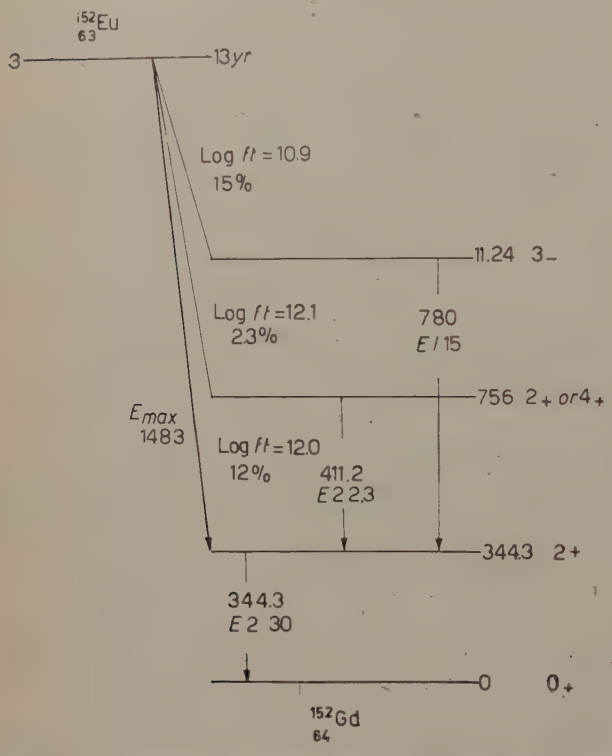

Fig. 1. – Partial decay-scheme of ^{152}Eu. For full decay-scheme see Fig. 1 of ref. ([6]).

The 1 483 keV β-group leads to the 1st excited state of ^{152}Gd at 344 keV which has character 2^+ and which deexcites by a pure $E2$ transition ([2]). The partial decay-scheme of 13 yr. ^{152}Eu decaying to levels in ^{152}Gd by β-emission is shown in Fig. 1. The present experiment aims at determining the β-γ directional correlation anisotropy, if any, in the 1 483 keV β → 344 keV γ-cascade of sequence $3^-(\beta)\cdot 2^+(\gamma)0^+$. An inspection of the decay scheme will reveal that this experiment can be carried out for β energies of 1 000 keV and above without any interference from other inner β-groups. The K-conversion coefficient and K/L ratio measurements on the 344 keV γ-ray ([1,2]) have established the character of the γ-ray to be a pure electric quadrupole type. A β-γ directional correlation experiment will unambiguously establish whether the β-decay is « unique » or « non-unique », since « unique » character of the transition will give an « unique » energy dependence of the anisotropy (see Eq. 4). Most of the non-unique first forbidden transitions have an allowed shape β-ray spectrum, which is explained by the fact that the shape correction factor is constant

([5]) M. Abraham, R. Kedjie and C. D. Jeffries: *Phys. Rev.*, **108**, 58 (1957); Manenkov, Prokhorov, Trukhliaev and Iakovlov: *Dokl. Akad. Nauk. S.S.S.R*, **112**, 623 (1957); Translation: *Sov. Phys. « Doklady »*, **2**, 64 (1957).

([6]) O. Nathan and S. Hultberg: *Nucl. Phys.*, **10**, 118 (1959).

in the ξ approximation ([7,8]) in which only the first non-vanishing term is kept in the expansion in descending powers of the Coulomb energy factor $\xi(=\alpha Z/2\varrho)$, where ϱ is the nuclear radius measured in units of the electron Compton wavelength. However, in cases where there is a « cancellation effect » or a « selection rule effect » ([9]) the above approximation seems to lose validity and deviations may occur. The observation of these deviations means detection of higher order contribution in the ξ-expansion. The observation of a β-γ directional correlation anisotropy is a means of detecting such deviations. Further, these cases offer valuable relations among the nuclear matrix elements.

Following KOTANI ([9]), one can state that the β-γ directional correlation is expressed by the relation

$$W'(\theta, E) = 1 + \varepsilon P_2(\cos\theta)\,, \tag{1}$$

where ε is the correlation coefficient, θ is the angle between β and γ-rays, $P_2(\cos\theta)$ is the Legendre's Polynomial and E is the kinetic energy of the β-particle. Equation (1) can also be expressed as

$$W(\theta, E) = 1 + a(E)\cos^2\theta\,, \tag{2}$$

where $a(E)$ is called the anisotropy.

Experimentally one determines the anisotropy $a(E)$, where

$$a(E) = \frac{W(\pi, E)}{W(\pi/2, E)} - 1 = \frac{\frac{3}{2}\varepsilon}{1 - \frac{1}{2}\varepsilon}\,. \tag{3}$$

where $W(\theta, E)$ is proportional to the β-γ coincidence rates. The correlation coefficient ε is predicted theoretically.

2. – Experimental procedure.

The source used for the present work was produced by bombarding enriched ^{151}Eu (92 per cent) with thermal neutrons in the Oak Ridge National Laboratory Reactor. The source was deposited on a 0.6 mg/cm^2 mylar foil over a diameter of about 3 mm. The foil was mounted on a thin perspex ring used as source holder. The source holder was carefully positioned at the centre of the angular correlation table. γ-rays ware detected by a 1 in. thick $\times 1\frac{1}{2}$ in. diam. NaI(Tl)

([7]) T. KOTANI and M. ROSS: *Phys. Rev. Lett.*, **1**, 140 (1958), eq. (5).
([8]) T. KOTANI and M. H. ROSS: *Phys. Rev.*, **113**, 622 (1959).
([9]) T. KOTANI: *Phys. Rev.* **114**, 795 (1959).

crystal placed at a distance of 7 cm from the source. The crystal was viewed by a Dumont 6292 photomultiplier. A 0.7 g/cm² lead absorber was used to stop all X-rays and low energy γ-rays which otherwise would unnecessarily load the electronic system. The crystal was also surrounded on the sides by 0.7 g/cm² lead sheet. Resolution for the 661 keV γ-ray of ^{137}Cs was about 8%. The γ-counter was used as the movable counter and the fixed γ-counting rate was constant to about 1% over all angles. The 344 keV photopeak was accepted in the γ-channel with a narrow window of 70 keV. To detect β-rays an anthracene crystal 1½ in. diam. × ¼ in. thick and a Dumont 6292 photomultiplier were used. This crystal was placed at a distance of 4.5 cm from the source and it was covered by a 3 mg/cm² aluminium foil in the front. Resolution for the 624 keV K-conversion electron line of ^{137}Cs was about 18% (with 80 keV window in the single channel analyser).

The electronic system consisted of a fast-slow coincidence arrangement with single channel pulse height analysers for energy selection. The equipment was capable of handling high counting rates. Blocking oscillators have been used as pluse-shapers in the fast coincidence circuit. A resolving time of $2 \cdot 10^{-8}$ s was used in the experiment and found to be very constant over long periods of time. Chance coincidences were measured from time to time by inserting a delay of 0.3 μs in either of the fast channels. The number of random coincidences was about 20% of the total number of coincidences in most of the measurements.

In the *integral anisotropy* measurement, coincidence rates were determined at several angular settings in the range 90° to 270° in steps of 15°. At each angle, a certain fixed number of coincidence counts was collected and the time taken for the same was noted. The individual channel counts accumulated during this time interval were also recorded. The latter enable us to determine the channel counting rates at every setting. The net coincidence rates were normalized to the channel rates to correct for any positioning error of the source and also small statistical drifts in the analysers. The corrected true coincidence counting rates for pairs of angles symmetric about 180° were averaged.

In the *differential anisotropy* measurements, readings were taken at 90°, 180° and 270° only. In this case, the number of coincidences collected in a fixed period of time was observed. Measurements have been made at eight β-energies: 950, 1000, 1050, 1100, 1150, 1250, 1300 and 1350 keV with a 80 keV window in the pulse analyser. The energy setting was varied randomly and the time devoted to collect coincidences was correspondingly adjusted. A normalization procedure similar to that adopted in the integral anisotropy measurements was followed in this case also.

A γ-γ coincidence background of the order of 10% of the gross coincidence rate was found to be present up to (1100) keV (see Fig. 2 and also the decay

scheme in Fig. 1 of ref. ([6])). This was measured by putting a 0.9 g/cm² Al absorber, which would stop all the β-rays, in front of the β-crystal. The γ-γ background coincidence rates and random coincidence rates were subtracted from the gross coincidence rates to give the genuine coincidence rates.

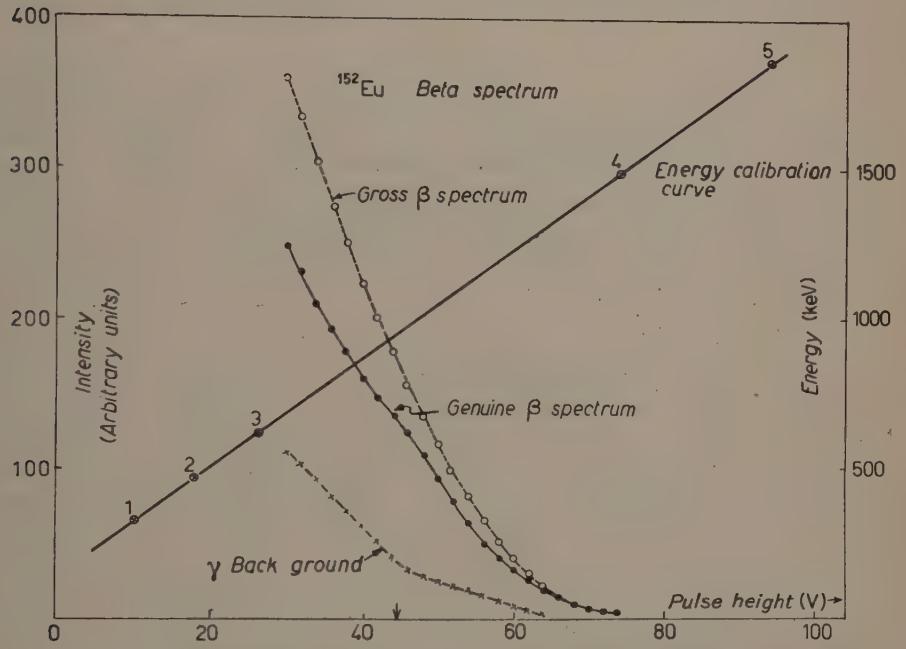

Fig. 2. – ^{152}Eu β-spectrum observed with an anthracene crystal spectrometer with 80 keV window in the analyser. The calibration points (1), (2), (3), (4) and (5) are described in the text.

For β-energy calibration, the following standards have been used:

1) 330 keV *K*-conversion line of the 411 keV γ-ray in ^{198}Au observed in coincidence with the 960 keV β-ray group.

2) 478 keV Compton edge of the 661 keV γ-ray in ^{137}Cs.

3) 624 keV *K*-conversion line of ^{137}Cs observed in coincidence with *K* X-ray.

4) 1 483 keV end point in the β-spectrum of ^{152}Eu.

5) 1 860 keV end point in the β-spectrum of ^{154}Eu.

The energy calibration is shown in Fig. 2.

The integral correlation data were analysed by Rose's method [10] to evaluate the values of the coefficients and the errors. The differential anisotropies were calculated from the 90° and 180° coincidence data. The errors shown are the statistical errors. Errors in chance measurements and γ-γ background coincidence measurements have also been taken into account.

The following corrections have been applied to the experimentally observed anisotropies:

i) *Geometrical correlations.* – For the geometry used in the experiments (β crystal 4.5%, γ crystal 1.8%), the correction for angular resolution is important. For the γ-counter, the correction was carried out using the correction tables of Rose [10]. The correction for the β-counter was calculated using the formulas of Rose [10] and Breitenberger [11]. The geometrical correction amounts to 15% of the anisotropy. Correction due to finite extension of the source has been neglected.

ii) *Correction for finite β energy resolution and backscattering.* – The line shape for the 624 keV K-conversion line was found to be very nearly Gaussian even with a 80 keV window. The resolution at all other energies were calculated assuming that the response of the phosphor-photomultiplier combination is statistical and that the line shapes are Gaussian. These corrections were carried out using the method due to Freedman *et al.* [12]. In the energy region of interest this correction is of the order of 1%.

iii) *Extra-nuclear field effects.* – The 344 keV level in ^{152}Gd is known to be vibrational in nature and hence one expects that it will have a life time less than 10^{-10} s. So it is very unlikely that there will be any perturbation due to extranuclear fields. Besides the γ-γ angular correlation coefficient of the (780 ÷ 344) keV cascade, as measured by Wiedenbeck [13], agrees well with the theoretical value for a $3^-(E1)2^+(E2)0^+$ γ-γ cascade where the 344 keV state enters as the intermediate state. Hence the contribution due to any extranuclear field effect is assumed to be negligible in the β-γ correlation also.

iv) *Inner bremsstrahlung effect.* – Inner bremsstrahlung of high energy β-rays may give rise to a photon associated with a β-particle having the remaining energy. This would cause a coincidence and may simulate a β-γ correlation if the energies are appropriate *e.g.* a photon with energy around 344 keV and a remaining β with energy above 950 keV. The relative number of I.B.

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[10] M. E. Rose: *Phys. Rev.* **91**, 610 (1953).

[11] E. Breitenberger: *Proc. Phys. Soc.* (*London*), **66** A, 846 (1953).

[12] M. S. Freedman, T. B. Novey, F. T. Porter and F. Wagner jr.: *Rev. Sci. Inst.*, **27**, 716 (1956).

[13] R. W. Lide and M. L. Wiedenbeck: *Phys. Rev.*, **113**, 840 (1959).

photons per β-decay is quite small and in our case only a fraction of the β-rays (with energy > 1300 keV) can cause a coincidence. Besides the angular correlation of I.B.-β as observed by NOVEY [14] is strongly peaked around 30° and falls off sharply and there is no I.B. at angle 180° to the betas. All these go in reducing the I.B.-β anisotropy such that it is very much within the errors of our anisotropy measurements and is therefore neglected.

3. – Results.

To check that there is no instrumental asymmetry the β-γ angular correlation in ^{60}Co was observed accepting all β-rays above 200 keV in the β-channel. The correlation function (uncorrected for geometry) obtained from a least

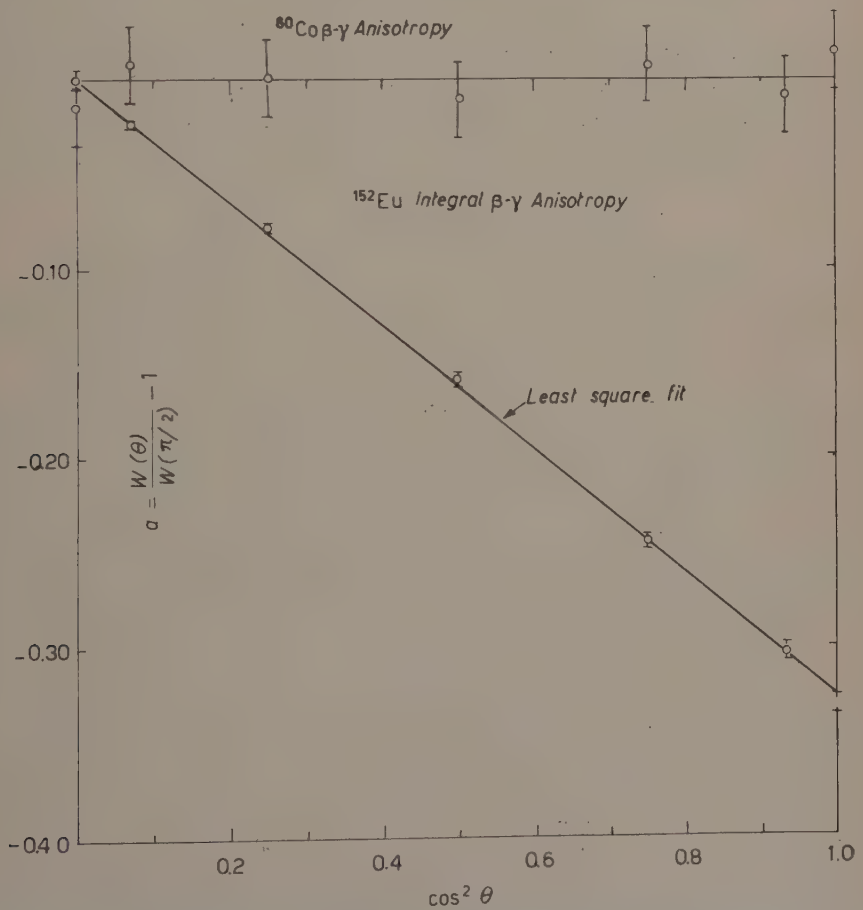

Fig. 3. – Integral β-γ anisotropy in ^{152}Eu with β-energy above 950 keV.

[14] T. B. NOVEY: *Phys. Rev.*, **89**, 672 (1953).

square fit ([10]) of the experimental results is

$$W'(\theta) = 1 + (0.007 \pm 0.008) P_2(\cos\theta) .$$

The experimental anisotropies are shown at the top of Fig. 3. Within the limits of experimental errors there is no observed anisotropy. It is thought however that ^{60}Co is not a very good standard for this purpose because the β-energy is very low and the experiment was done in air. The correlation could be affected by the scattering of betas in air. In ^{152}Eu experiments, however, this effect is assumed to be negligible as our β-energy is above 950 keV.

3·1. *Integral anisotropy.* – The integral β-γ anisotropy in ^{152}Eu was observed accepting β-particles of energy > 950 keV. In ^{152}Eu there is an inner β group (see Fig. 1) with an end point energy of 1 070 keV whose intensity is about 6 times less than that of 1 483 keV. The contribution due to this group in the integral correlation data will be small above 950 keV compared with the 1 483 keV β-group (less than 1%) and is therefore neglected.

The observed correlation function, using only the $P_2(\cos\theta)$ term is

$$W'(\theta) = 1 - (0.244 \pm 0.003) P_2(\cos\theta) .$$

A least square fit taking the $P_4(\cos\theta)$ term also into account gives

$$W'(\theta) = 1 - (0.240 \pm 0.003) P_2(\cos\theta) - (0.008 \pm 0.003) P_4(\cos\theta) .$$

The small value of the coefficient in the $P_4(\cos\theta)$ term as compared to that in the $P_2(\cos\theta)$ term indicates that the parity of ^{152}Eu is odd ([9]). After correcting for geometry, the correlation function is

$$W'(\theta) = 1 - (0.289 \pm 0.003) P_2(\cos\theta) .$$

The experimental anisotropy

$$a(\theta) = \frac{W(\theta)}{W(\pi/2)} - 1$$

is plotted as a function of $\cos^2\theta$ in Fig. 3.

The observed anisotropy is

$$a = \frac{W(\pi)}{W(\pi/2)} - 1 = -0.326 \pm 0.004 .$$

The value after geometrical correction is

$$a = -0.379 \pm 0.004 .$$

3·2. *Differential anisotropy.* – These experiments were done at the energies mentioned earlier with a 80 keV window in the analyser. The observed differential anisotropies, after making all corrections as mentioned above are plotted as a function of β-energy in Figs. 4, 5 and 6 (*).

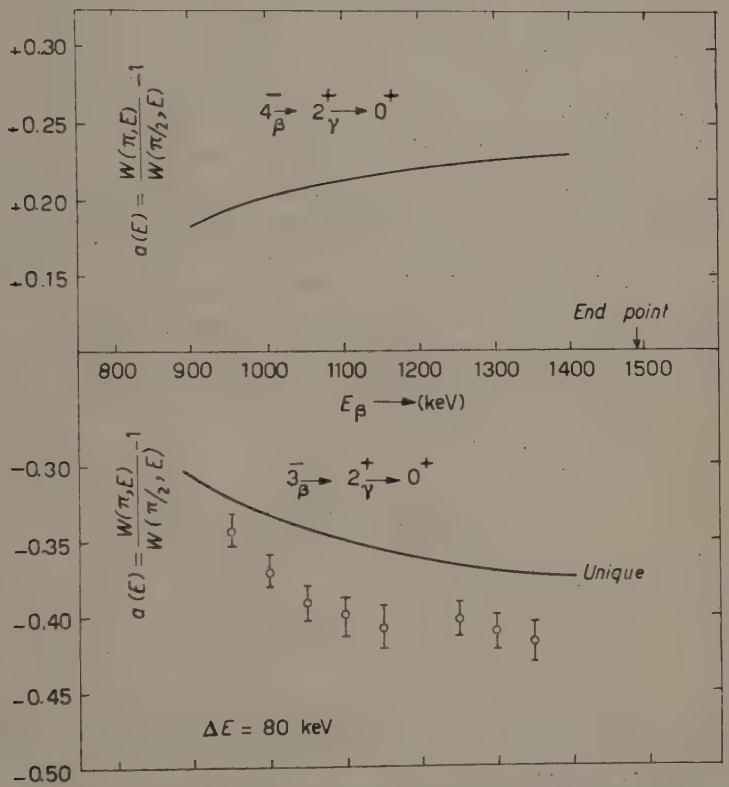

Fig. 4. – Differential β-γ anisotropy in ^{152}Eu. The vertical flags of the experimental points represent statistical errors. The theoretical energy dependence of the anisotropy for cascades $4^-(\beta)2^+(\gamma)0^+$ and $3^-(\beta)2^+(\gamma)0^+$ assuming unique transitions in both cases are also shown.

Integration of these anisotropies over the beta spectrum in the energy range > 950 keV gives $a = -0.377$ in comparison with $a = -0.379 \pm 0.004$ obtained in the integral correlation experiment.

(*) It has been recently brought to our notice that a preliminary measurement has been made on the β-γ angular correlation at about 1 100 keV β-energy of the 1 483 keV group with 344 keV γ-ray (H. DULANEY, L. D. WYLY and C. H. BRADEN: *Bull. Am. Phys. Soc.*, **4**, no. 6, 391 (1959)) and they report a negative anisotropy of approximately 0.35. Our experimental anisotropy at 1 100 keV, without any correction is -0.340 ± 0.014.

4. – Discussion.

Recently KOTANI and ROSS (8), KOTANI (9), MORITA and MORITA (15) have discussed in great detail the theory of first forbidden β-transitions. In interpreting our β-γ directional correlation results we shall follow KOTANI (9), who has given explicit expressions for the β-γ correlation coefficient ε and the shape correction factor $C(W)$ for the β-γ cascade of sequence $3^-(\beta)2^+(\gamma)0^+$. We shall assume $V-A$ theory, the two component neutrino theory ($C_A = C'_A$ and $C_V = C'_V$) and the validity of time-reversal invariance in β-decay.

KOTANI (9) argues that in order to explain the abnormally large log (ft) value of about 12 for the 1 483 keV β-transition one may assume a large selection rule effect due to K-forbiddenness (16). Such a selection rule effect has a tendency to inhibit the first forbidden matrix elements of rank $\lambda = 1$ like $\int \boldsymbol{\sigma} \times \boldsymbol{r}$, $\int \boldsymbol{r}$ and $\int \boldsymbol{\alpha}$ in comparison to the matrix element of rank $\lambda = 2$ like $\int B_{ij}$.

Supposing that the selection rule effect is complete we shall have only the B_{ij} matrix element present in this decay. In such a case the β-γ directional correlation coefficient ε will be given by (8)

$$\varepsilon = -\frac{2}{7}\frac{\lambda_1 p^2}{[(W_0 - W)^2 + \lambda_1 p^2]} \tag{4}$$

and the spectral shape is unique

$$C(W) = \frac{1}{12}[(W_0 - W)^2 + \lambda_1 p^2]\,. \tag{5}$$

Here λ_1 and λ_2 of eq. 8 contain Coulomb corrections of $(\alpha Z W/p)^2$ and are tabulated in Tables I and II of ref. (8).

In Fig. 4 the experimental differential anisotropies $a(E)$ are plotted as a function of electron energy E. The observed anisotropies have the correct sign but not the magnitude for a pure B_{ij} type of β-transition in the $3^-(\beta)2^+(\gamma)0^+$ sequence. In Fig. 4 is also shown the expected energy dependence of β-γ anisotropy for a sequence $4^-(\beta)2^+(\gamma)0^+$ sequence as given by the equation

$$\varepsilon = \frac{1}{7}\frac{\lambda_1 p^2}{[(W_0 - W)^2 + \lambda_1 p^2]}, \tag{6}$$

and shape is unique as given by equation (5).

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(15) M. MORITA and R. S. MORITA: *Phys. Rev.*, **109**, 2048 (1958).

(16) G. ALAGA, K. ALDER, A. BOHR and B. R. MOTTELSON: *Kgl. Dan. Videnskab. Selskab. Mat.-fys. Medd.*, **29**, no. 9 (1959).

The sign and magnitude of the observed β-γ anisotropies rules out a 4- character to the ^{152}Eu ground state ([1,2]).

An examination of Fig. 4 suggests that in this β-transition we have to consider the effect of matrix elements of rank $\lambda = 1$ in addition to B_{ij}. KOTANI ([9]) points out that the quantitative results for each observable in this β-decay may be characterized by the so-called « modified B_{ij} approximation ». The value of $\xi(=\alpha Z/2\varrho)$ is equal to 13.8 in this transition as $Z = 63$ and $\varrho = (1.2 \cdot 10^{-13} A^{\frac{1}{3}})/(\hbar/mc)$ and $\alpha = 1/137$. ξ is much greater than $W_0 = 3.902$. Further, if a selection rule effect of the type of K-forbiddenness operates in this decay, it will increase the contribution of B_{ij} in comparison to other first forbidden matrix elements. Hence one is perhaps justified that the modified B_{ij} approximation is valid for this β-transition. In the notations of KOTANI ([9]) the following matrix elements enter into this transition:

$$z = 1, \qquad Y = \xi' y, \qquad u = x = 0, \tag{7}$$

where

$$\eta u = C_A \int i\boldsymbol{\sigma} \times \boldsymbol{r}, \qquad \eta x = -C_V \int \boldsymbol{r}.$$

$$\eta \xi' y = -C_V \int i\boldsymbol{\alpha} \qquad \text{rank } \lambda = 1$$

$$\eta z = C_A \int B_{ij} \qquad \text{rank } \lambda = 2$$

and $\xi' = \xi\lambda$ ([7]) with $\lambda = 1$ or 2.

In this modified B_{ij} approximation, the β-γ anisotropy co-efficient is given by KOTANI ([9])

$$\varepsilon = -(p^2/W)\,\frac{(1/7)\lambda_2 Y + (1/42)\lambda_1 W}{C(W)}, \tag{8}$$

where the shape correction factor to the β-spectrum is given by

$$C(W) = Y^2 + \frac{1}{12}\left[(W_0 - W)^2 + \lambda_1 p^2\right]. \tag{9}$$

An expression for the anisotropy $a(W)$ can be obtained by using equation (3). In eq. (8) Y appears quadratically, so for each value of ε there are two values of Y. Comparison of eq. (9) with the experimental shape factor ([4]) may decide which value of Y is appropriate to this decay.

The theoretical expression for $a(W)$ for any particular β-energy if equated with the experimental value and solved for Y, will give two values, say Y_1

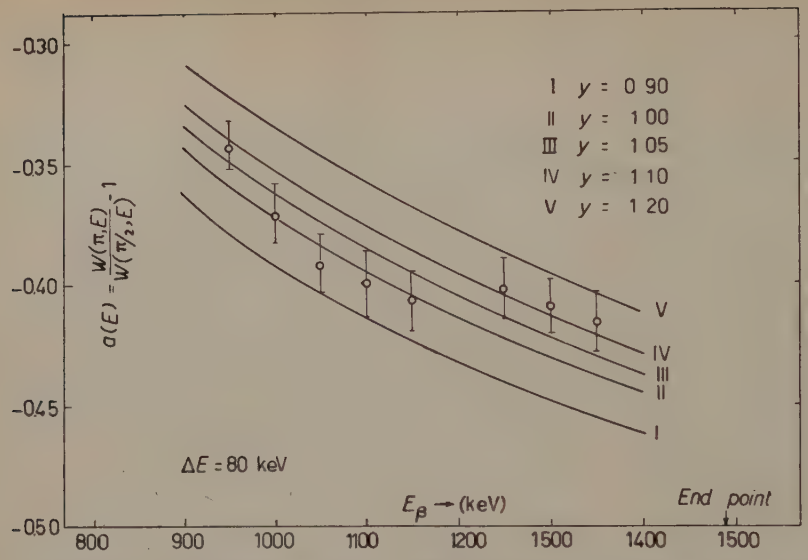

Fig. 5. – Differential β-γ anisotropy data in ^{152}Eu fitted with different values of matrix element ratio Y in the range $0.90 \div 1.20$.

and Y_2. There will, in general, be some ranges of values for Y_1 and Y_2 due to scatter of experimental points. The fitting of the observed energy dependence of β-γ anisotropy can be done by putting in eq. (8) various values of Y_1 and Y_2 from their respective ranges. Figs. 5 and 6 show such fittings. It will

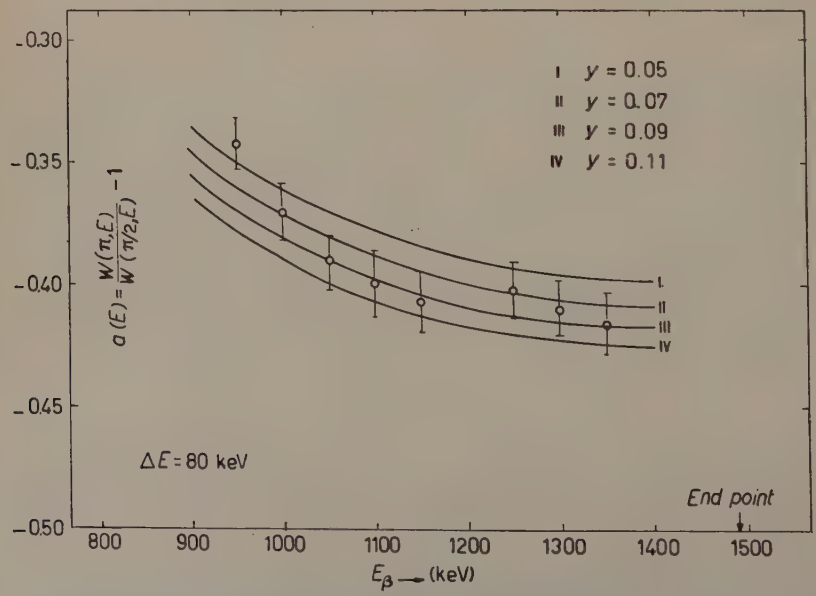

Fig. 6. – Differential β-γ anisotropy data in ^{152}Eu fitted with different values of matrix element ratio Y in the range $0.05 \div 0.11$.

be seen from these figures that $Y \simeq 1.05$ and $Y \simeq 0.09$ give the best fit to the observed anisotropy.

In Fig. 7 we have plotted $\varepsilon(p^2/W)^{-1}$ as a function of β-energy E. KOTANI (9) points out that such a plot brings out clearly the effect of deviations from the ξ-approximation since in the ξ-approximation ε has energy dependence proportional to (p^2/W) (see eq. (11) of ref. (8)). Theoretical plots for $Y = 1.05$ and $Y = 0.09$ calculated from eq. (8) are shown; also such a plot for the « unique » (B_{ij} only) transition for a $3^-(\beta)2^+(\gamma)0^+$ sequence has been shown Our results establish that the 1 483 keV β-transition is a « non-unique » one.

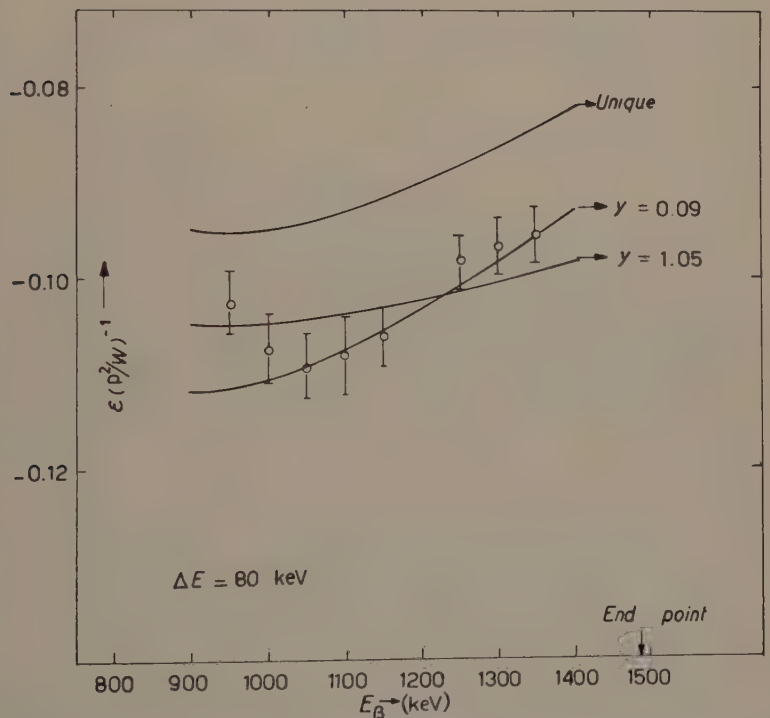

Fig. 7. – The quantity $\varepsilon(p^2/W)^{-1}$ plotted against the β-particle energy E. The three solid lines represent the theoretical value corresponding to unique transition, $Y = 0.09$ and $Y = 1.05$ respectively.

In Fig. 8 we have plotted the numerical values of the experimental shape correction factor $C(E)$ for β-energies from above 1 100 keV as given recently by LANGER *et al.* (4)

$$C(E) = (W_0 - W)^2 + 0.791p^2 + 5 \pm 2 \,. \tag{10}$$

The shaded portion shows the plot of the above equation within its limits where W_0 is taken to be 3.902 corresponding to maximum energy of the β-particle of 1483 keV. The theoretical $C(W)$ as given by eq. (9) is plotted for $Y = 1.05$ and $Y = 0.09$ and these values are normalized with the mean experimental values of $C(E)$ at two energies; *e.g.*, at 1100 keV and 1300 keV.

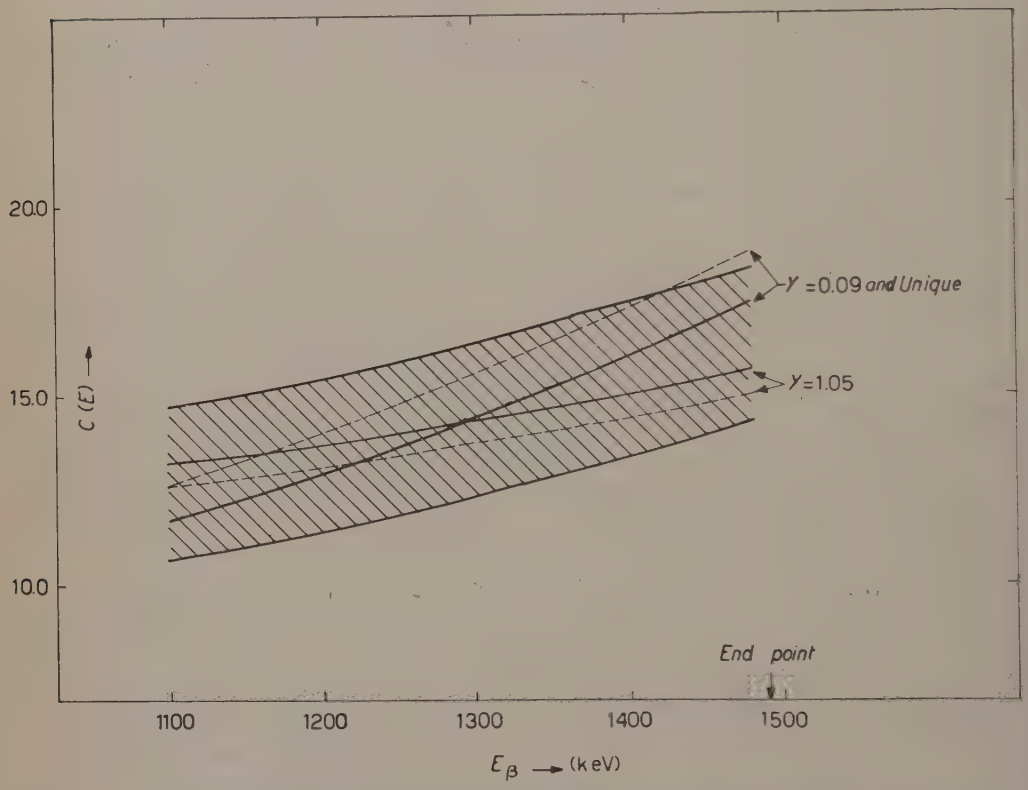

Fig. 8. – The shaded portion represents the experimental shape correction factor within its limits as given by LANGER *et al.* (4) eq. (10). The two dotted lines correspond to the two values of Y normalized at $E_\beta = 1100$ keV. The solid lines correspond to the two values of Y normalized at $E_\beta = 1300$ keV.

The unique shape correction factor eq. (5) which has been observed earlier (1,2), is also plotted, which is, however, indistinguishable from the plot of eq. (9) with $Y \simeq 0.09$; *i.e.* even though this β-transition is a non-unique one the shape may be very much like that of a « unique » one. However, the slope of the graph corresponding to $Y \simeq 1.05$ perhaps agrees better with the experimental shape of LANGER *et al.* (4).

5. – Conclusion.

Thus, we conclude that the 1 483 keV β-transition in 13-year ^{152}Eu is non-unique; the energy dependence of the observed β-γ anisotropy above β-energy of 1 MeV agrees, within the limitations of the « modified B_{ij} approximation », to a sequence $3^-(\beta)2^+(\gamma)0^+$ with the matrix element ratio $Y \simeq 1.05$ or 0.09; the former value of Y reproduces closely the spectral shape correction factor experimentally ([4]) determined.

* * *

We like to thank Drs. K. KUMAR and V. GUPTA for helpful discussions

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RIASSUNTO (*)

Si è misurata, in funzione della energia β al di sopra di 950 keV, la correlazione direzionale fra il primo gruppo proibito di 1483 keV e la transizione γ $E2$ di 344 keV nel decadimento del ^{152}Eu (13 anni). I dati della correlazione integrale β-γ per energie β al di sopra di 950 keV può essere rappresentata con una funzione di correlazione direzionale

$$W(\theta) = 1 - (0.379 \pm 0.004) \cos^2 \theta \,.$$

L'anisotropia è negativa e si riscontra che cresce con l'energia β, il suo valore è -0.416 ± 0.012 a 1350 keV, energia massima alla quale vennero eseguite misurazioni. La dipendenza dell'anisotropia dall'energia esclude il carattere unico della transizione β. La dipendenza osservata dell'anisotropia è rappresentabile con la teoria (V—A) nella approssimazione modificata B_{ij} che da due valori del rapporto fra gli elementi della matrice $Y = -(C_V/C_A)(\int i\alpha/\int B_{ij}) \approx 1.05$ o 0.09. Il primo valore di Y, comunque, riproduce bene la forma dello spettro osservato di recente per il gruppo β di 1483 keV.

(*) *Traduzione a cura della Redazione.*

Considerazioni sui limiti di applicabilità del metodo degli atomi nelle molecole.

E. Abate

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Gruppo di Fisica dei Solidi del C.N.R. - Milano

(ricevuto il 7 Gennaio 1960)

Il modo più semplice di affrontare i problemi molecolari è quello delle approssimazioni « ad elettroni singoli ». Si scelgono come funzioni base, o di approssimazione zero, prodotti antisimmetrizzati di funzioni monoelettroniche e si esprime un particolare stato molecolare ψ_k come combinazione lineare delle funzioni F scelte come base:

$$\psi_k = \sum_b F_b c_{bk} , \tag{1}$$

(due tipi di funzioni base frequentemente usate sono quelle *ASMO* e quelle *HLSP*).

I coefficienti c_{bk} vengono determinati con un metodo variazionale e l'energia E_k dello stato ψ_k si trova come la soluzione più bassa dell'equazione secolare

$$|H - E_k M| = 0 , \tag{2}$$

dove H ed M sono rispettivamente la matrice hamiltoniana e la matrice di sovrapposizione nello schema scelto. In particolare nello schema *ASMO* la matrice M si riduce alla matrice unità ed E_k è l'autovalore più basso della matrice H.

Con procedimenti di questo tipo si possono calcolare le energie elettroniche che competono ai diversi stati di una molecola, e quindi le energie di eccitazione, che sono confrontabili con le frequenze sperimentali. I risultati, per quanto riguarda le energie di eccitazione, non sono in genere molto soddisfacenti per effetto delle cosiddette « energie di correlazione » e il divario tra i valori calcolati e quelli osservati è spesso notevole (per il litio si trovano discordanze di circa il 40%).

Il metodo degli atomi nelle molecole [1,2] si propone di ridurre tale divario tenendo conto in modo corretto delle energie di correlazione nei sistemi di atomi separati.

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[1] W. Moffitt: *Proc. Roy. Soc.*, A **210**, 224 (1951).
[2] W. Moffitt: *Proc. Roy. Soc.*, A **210**, 245 (1951).

Si consideri l'operatore hamiltoniano come somma di due parti:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{V}, \tag{3}$$

di cui $\mathscr{H}_0$ è l'hamiltoniano relativo al sistema di atomi considerati a distanza infinita e $\mathscr{V}$ l'operatore corrispondente all'interazione degli atomi per la formazione della molecola. La matrice H si può scrivere allora:

$$H = \tfrac{1}{2}(MW + WM) + \tfrac{1}{2}(V + V^{\dagger}), \tag{4}$$

dove W e V sono le matrici corrispondenti a $\mathscr{H}_0$ e $\mathscr{V}$ nella base scelta.

La correzione di Moffitt consiste nel sostituire alla matrice W teorica la matrice diagonale che ha come elementi le energie « corrette » dei prodotti atomici o ionici di dissociazione negli « stati di valenza » corrispondenti alle funzioni molecolari considerate. Tali energie « corrette » si possono ricavare o da calcoli atomici completi o da dati sperimentali. Applicando la correzione nello schema *ASMO* (in cui $M = 1$) per gli elementi di matrice corretti si ottiene la formula:

$$H = H^{\text{teorica}} - (W^{\text{teorica}} - W^{\text{corretta}}). \tag{5}$$

Nello schema *HLSP* invece la correzione diventa:

$$H = H^{t} + \tfrac{1}{2}\{M^{t}(W^{c} - W^{t}) + (W^{c} - W^{t})M^{t}\}. \tag{6}$$

Questa seconda forma, proposta nel secondo articolo di Moffitt sull'argomento [2] è preferibile alla (5) perchè la correzione di Moffitt è più nello spirito dell'approssimazione *HLSP* che in quello dell'approssimazione *ASMO*. La correzione, per ogni funzione molecolare, infatti, è fatta partendo da considerazioni sui suoi prodotti di dissociazione, e, nello schema *ASMO*, i prodotti di dissociazione non descrivono completamente il contenuto atomico di una funzione molecolare. D'altra parte è più comodo usare come base le funzioni *ASMO* data la maggiore ricchezza di integrali tabulati; l'applicazione della (6) implica quindi il calcolo degli elementi della matrice T di trasformazione dalle funzioni *HLSP* alle *ASMO*.

Applicando il metodo qui richiamato alle molecole di ossigeno, di etilene, di alcuni composti aromatici [1-5], Moffitt e collaboratori hanno trovato risultati soddisfacenti, cioè un accordo con i dati sperimentali molto migliore che nel caso di calcoli puramente teorici. Invece le applicazioni fatte da Pauncz [6] e Hurley [7] alla molecola di idrogeno e da Rahman [8] alla molecola di litio, cioè a molecole molto più semplici, portano a considerazioni piuttosto negative sull'utilità del metodo.

Più precisamente, Hurley e Pauncz trovano miglioramenti irrilevanti rispetto ai risultati ottenuti nello schema *HLSP* mentre Rahman trova addirittura risultati peggiori di quelli teorici.

Si è pensato quindi di fare una nuova applicazione del metodo di Moffitt alla molecola di litio, dato anche che Rahman applicava formule di correzione, per gli

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([3]) W. Moffitt and J. Scanlan: *Proc. Roy. Soc.*, A **218**, 464 (1953).
([4]) W. Moffitt: *Proc. Roy. Soc.*, A **218**, 486 (1953).
([5]) W. Moffitt and J. Scanlan: *Proc. Roy. Soc.*, A **220**, 530 (1953).
([6]) A. C. Hurley: *Proc. Phys. Soc.*, A **68**, 149 (1955).
([7]) R. Pauncz: *Acta Physica*, **4/3**, 237 (1954).
([8]) A. Rahman: *Physica*, **20**, 623 (1954).

elementi della matrice H, basate sull'idea di Moffitt, ma non identiche a quelle qui riportate, e cui Fieschi [9] aveva applicato la correzione nella forma (5).

Poichè la molecola di litio è la più semplice dopo quella di idrogeno, i risultati di questo calcolo dovrebbero dare indicazioni sui limiti di applicabilità del metodo.

Il calcolo è stato semplificato supponendo gli elettroni 1s nel nucleo, semplificazione già adottata da Rahman, che permette di scrivere le funzioni d'onda come prodotti antisimmetrizzati di due sole funzioni monoelettroniche. Si sono calcolate per varie distanze internucleari le energie elettroniche dei tre stati $^1\Sigma_g^+$ (fondamentale), $^1\Sigma_u^+$ e $^1\Pi_u$ e le energie di eccitazione per le transizioni $^1\Sigma_u^+$-$^1\Sigma_g^+$, $^1\Pi_u$-$^1\Sigma_g^+$.

I calcoli teorici sono stati presi dal lavoro di Fieschi. L'ultima colonna della tabella degli stati atomici e ionici del litio è stata corretta perchè mancava il termine di energia cinetica; riportiamo qui la tabella corretta (Tab. III, p. 201):

Tabella I. – *Energia degli stati di interesse di* Li^+, Li, Li^-.

Stato atomico o ionico	Configurazione elettronica	Energia sperim. (in u.a.)	Energia teor.
Li^+ (S)	(si trascurano gli el. K)	0	0
Li (^2S)	$2s$	— 0.1981	— 0.208333
Li (^2P)	$2p$	— 0.1302	— 0.125000
Li^- (^1S)	$(2s)^2$	— 0.1981	— 0.235026
Li^- (^1P)	$(2s, 2p)$	— 0.1981	— 0.111545
Li^- (^1D)	$(2p)^2$	— 0.1981	— 0.064844
Li^- $(^1S')$	$(2p)^2$	— 0.1981	— 0.033203

La correzione di Moffitt è stata applicata nella forma (6) ed è stato ripetuto il calcolo nella forma (5); i risultati ottenuti nei due casi non sono sensibilmente diversi.

Nella Tab. II sono riportati i risultati ottenuti per le energie di eccitazione applicando la correzione (6), insieme a quelli ricavati da dati sperimentali e a quelli ottenuti con il calcolo puramente teorico nello schema *ASMO*.

Tabella II. – *Energie di eccitazione* (*in* u.a.).

Transizione		α		
		2.5	2.75	3
$^1\Pi_u$ - $^1\Sigma_g^+$	*ASMO*	0.1593	0.1520	0.1444
	Moffitt	0.0692	0.0798	0.0861
	sperim.	0.0952	0.0988	0.0891
$^1\Sigma_u^+$ - $^1\Sigma_g^+$	*ASMO*	0.1175	0.1166	0.1129
	Moffitt	lo stato $^1\Sigma_u^+$ risulta più basso del $^1\Sigma_g^+$		
	sperim.	0.0684	0.0634	0.0593

($\alpha = \delta R$, dove $\delta = 0.5$ e R è la distanza internucleare in u.a.)

[9] R. Fieschi: *Nuovo Cimento*, **6**, 197 (1957).

Dalla tabella risulta che per la transizione $^1\Pi_u$-$^1\Sigma_g^+$ la correzione migliora notevolmente l'accordo dei valori calcolati con quelli sperimentali, mentre per la transizione $^1\Sigma_u^+$-$^1\Sigma_g^+$ la correzione agisce in modo del tutto negativo, portando addirittura all'inversione dell'ordine dei due stati. Applicato alla molecola di litio quindi il metodo degli atomi nelle molecole sembra del tutto inefficiente (*).

Sembra di poter concludere che il metodo di Moffitt non è applicabile a molecole semplici, per le quali le energie di correlazione non sono più rilevanti nei prodotti di dissociazione che nella molecola.

Si può inoltre osservare che una limitazione comune al metodo di Moffitt e ad altri metodi semi-empirici più elaborati ([10,11]) è la rilevante incertezza che si incontra nella determinazione delle energie degli ioni negativi.

* * *

Desidero ringraziare il Prof. R. Fieschi per avermi suggerito questo calcolo.

(*) Anche Arai *et al.* ([10]) applicando la correzione semi-empirica di Moffitt al litio e facendo il calcolo con funzioni d'onda a 6 elettroni (togliendo cioè l'approssimazione degli elettroni 1*s* nel nucleo) hanno trovato risultati che sono più in disaccordo con i dati sperimentali dei risultati puramente teorici.

([10]) T. Arai: *Journ. Chem. Phys.*, **26**, 435 (1957).

([11]) A. C. Hurley: *Proc. Phys. Soc.*, A **69**, 49 (1956).

The Electronic Decay of Λ^0 Derived Indirectly by K-Meson Couplings.

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(ricevuto il 29 Gennaio 1960)

The great success which universal vector-axialvector (V—A) couplings brought to the interactions of leptons and nucleons suggested that it be extended also to the hyperons. The simplest case is here the decay $\Lambda \to p + e^- + \bar{\nu}$. Experimentally two examples have already been observed, which equals 0.2% [1] of the total Λ decay rate ($W_{\Lambda\text{tot}al} = 3.8 \cdot 10^9\ s^{-1}$) [2]. A calculation with UFI gives 1.6% [3] of the total decay rate. Owing to this discrepancy it is of interest to study other decay mechanisms.

The Λ_{e3} decay can also be derived indirectly by K-meson couplings. Although the K_{e2} decay $K^- \to e^- + \bar{\nu}$ has not yet been observed, its existence is very probable [4]. Its effective coupling constant c can be calculated from $K_{\mu 2}$ decay with the effective interaction density

$$\mathscr{H}'_{\text{eff}} = c\bar{\psi}_l \gamma_\sigma (1 + \gamma_5)\psi_\nu \partial^\sigma \varphi_K \,. \qquad (l = e^- \text{ or } \mu^-).$$

But the graph in Fig. 1 with $\mathscr{H}_K = g_K \psi_\Lambda \gamma'_5 \psi_p \varphi_K$ in 1 and $\mathscr{H}'_{\text{eff}}$ in 2 gives a contribution which is by many orders too low.

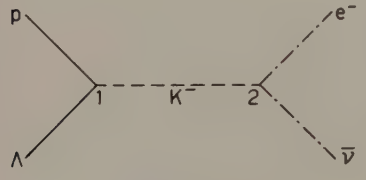

Fig. 1.

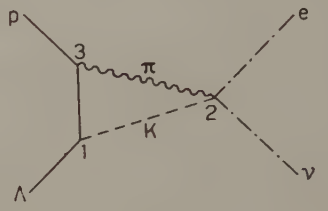

Fig. 2.

Other leptonic K decays are: $K^- \to \pi^0 + e^- + \bar{\nu}$ and $K^0 \to \pi^+ + e^- + \bar{\nu}$. They lead by means of the graph in Fig. 2 to Λ_{e3} decay. Herein the well known strong inter-

(1) F. S. Crawford, M. Cresti, M. L. Good, G. R. Kalbfleisch, M. L. Stenenson, H. K. Ticho: *Phys. Rev. Lett.*, **1**, 377 (1958); P. Nordin, J. Orear, L. Reed, A. H. Rosenfeld, F. T. Solmitz, H. D. Taft, R. D. Tripp: *Phys. Rev. Lett.*, **1**, 380 (1958); D. A. Glaser: *Strange particle decays*. Conf. on High Energy Physics (Kiew, 1959).

(2) D. A. Glaser, M. L. Good and D. R. O. Morrison: *Annual Intern. Conf. on High Energy Physics at CERN* (1958), p. 270.

(3) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 380 (1958); S. Okubo, R. E. Marshak ayd E. C. G. Sudarshan: *Phys. Rev.*, **113**, 944 (1959).

(4) R. H. Dalitz: *Rev. Mod. Phys.*, **31**, 823 (1959).

actions $\mathscr{H}_\pi = g_\pi \psi_p \gamma_5 \psi_n \varphi_\pi$ of the π-meson and $\mathscr{H}_K$ of the K-meson operate in 1 and 3. For the weak interactions in 2 we substitute the effective interaction density

$$\mathscr{H}_{\text{eff}} = f m_e \bar{\psi}_e(x)(1+\gamma_5)\psi_\nu(x)\,\varphi_\pi(x)\varphi_K(x) + g\bar{\psi}_e(x)\gamma_\lambda(1+\gamma_5)\psi_\nu(x)\,\varphi_\pi(x)\,\partial^\lambda\varphi_K(x)\,,$$

which is most convenient for calculating the K_{e3} decay rate in the K-meson rest system. It is also sufficiently general since in first order perturbation theory we can transform terms of the shape $\bar{\psi}_e\gamma_\lambda(1+\gamma_5)\psi_\nu\varphi_K\partial^\lambda\varphi_\pi(x)$ into the terms contained in $\mathscr{H}_{\text{eff}}$. This is still true if the bosons are virtual.

One also obtains this expression by calculating the K_{e3} decay with $V-A$ theory and nucleon-Λ loops ([5,6]). There the effective coupling constants f and g turn out to be of the same order of magnitude. The calculation of the decay rate gives all terms which contain f by a factor $(m_e/m_K)^2$ smaller than those containing g. Neglecting the f terms we get

$$W = g^2\frac{1}{(2\pi)^3}\frac{m_K}{4}\left[\frac{1}{3}p_\pi'^3 p_\pi^{0\prime} - m_\pi^2\left(p_\pi' p_\pi^{0\prime} + \frac{m_\pi^2}{2}\ln\frac{m_\pi}{m_K}\right)\right]$$

wherein

$$p_\pi' = \frac{m_K^2 - m_\pi^2}{2m_K} = 447.2\ m_e\,,$$

means the maximum momentum and

$$p_\pi^{0\prime} = \frac{m_K^2 + m_\pi^2}{2m_K} = 519.5\ m_e\,,$$

means the maximum energy of the π-meson in the K-meson rest system.

If we substitute $W(K^- \to \pi^0 + e^- + \bar{\nu}) = 3.42\cdot 10^6\ s^{-1}$ ([7]) we obtain $g = 0.93\cdot 10^{-33}\ cm^2$ or in units $\hbar = c = 1$, $g m_N^2 = 0.21\cdot 10^{-5}$ which is about the third part of the universal Fermi coupling constant $g_f m_N^2 = 0.714\cdot 10^{-5}$.

For the K_{e3} decay the decay rate is yet not very well known experimentally ([7,8]). Hence we use its theoretical value gained from isotopic spin selection rules ([4]). It is twice the K_{e3} decay rate which gives the $\sqrt{2}$-fold coupling constant.

The S-matrix element to the graph in Fig. 2 in momentum space is worked out to

$$S = -i\frac{g_\pi g_K}{4(2\pi)^4}\{\bar{w}_p w_\Lambda \bar{u}_e(1+\gamma_5)u_\nu m_e(fF - gG) +$$
$$+ gH\bar{w}_p w_\Lambda \bar{u}_e\gamma_0(1+\gamma_5)u_\nu - gR\bar{w}_p\gamma^\mu w_\Lambda \bar{u}_e\gamma_\mu(1+\gamma_5)u_\nu\}\delta(p_\Lambda - p - p_e - p_\nu)\,,$$

F, G, H are complicated Feynman integrals from which we have separated a factor $-i\pi^2$. They change by about 10% of their values as the momentum transfer

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([5]) F. Zachariasen: *Phys. Rev.*, **110**, 1481 (1958).
([6]) A. Fujii and M. Kawaguchi: *Phys. Rev.*, **113**, 1481 (1959).
([7]) M. Gell-Mann and A. H. Rosenfeld: *Ann. Rev. of Nucl. Sci.*, **7**, 407 (1957).
([8]) M. Bardon, K. Lande, L. M. Lederman and W. Chinowsky: *Ann. Phys.*, **5**, 156 (1958); F. S. Crawford, M. Cresti, R. L. Douglas, M. L. Good, G. R. Kalbfleish and M. L. Stevenson: *Phys. Rev. Lett.*, **2**, 361 (1959).

varies, but we will neglect this dependence and calculate with their values for mean energies. F, G, H, do not depend on the «cut off» λ. Only R contains λ logarithmically.

If we assume pseudovector couplings instead of the pseudoscalar couplings in 1 or 3, for instance

$$H'_K = \frac{g_K}{m_\Lambda + m} \overline{\psi}_Q \gamma_\lambda \gamma_5 \psi_\Lambda \hat{c}^\lambda \varphi_K ,$$

the shape of the matrix element will not change; we only obtain other quantities F', G', H' instead of F, G, H. We will only need the numerical values of

$$H = -0.286 , \qquad H' = 0.428 \qquad \text{and} \qquad R = -\frac{1}{2}\left(\ln\frac{\lambda}{m} + 0.077\right).$$

We obtain for the decay rate omitting terms which are smaller by factors m_e/m_Λ or even $(m_e/m_\Lambda)^2$:

$$W = \frac{1}{(2\pi)^5}\frac{g_\pi^2}{4\pi}\frac{g_K^2}{4\pi}\frac{g^2}{4}\left\{\frac{2}{3}H^2\frac{p'^5}{5} + \left[\frac{2}{3}mH^2 - \frac{4}{3}(m_\Lambda + m)HR\right]\frac{1}{8}\left[2p'^3p^{0\prime} - 3m^2\left(p'p'_0 - m^2\ln\frac{m_\Lambda}{m}\right)\right] + \right.$$
$$\left. + R^2 m_\Lambda\left[\frac{2}{3}(p'_0 + 2m)p'^3 - m(2p'_0 + m)\left(p'p'_0 - m^2\ln\frac{m_\Lambda}{m}\right)\right]\right\},$$

wherein

$$p' = \frac{m_\Lambda^2 - m^2}{2m_\Lambda} = 318.91\ m_e ,$$

means the maximum momentum and

$$p'_0 = \frac{m_\Lambda^2 + m^2}{2m_\Lambda} = 1\,863.59\ m_e$$

means the maximum energy of the proton in the Λ rest system.

If we substitute $g_\pi^2/4\pi = 13.5$, $g_K^2/4\pi = 1$ and carry out the numerical evaluation we obtain for various values of the covariant cut off λ the decay probabilities in Table I.

TABLE I. – $W(\Lambda \to p + e^- + \bar{\nu})\ s^{-1}$.

cut off λ	experimental $7.6 \cdot 10^6$	
	theoretical for virtual K^--mesons with pseudovector | pseudoscalar K-meson couplings in 1	
5 m	$2.04 \cdot 10^5$	$5.57 \cdot 10^4$
10 m	$3.03 \cdot 10^5$	$9.66 \cdot 10^4$
100 m	$8.99 \cdot 10^5$	$4.92 \cdot 10^5$

Decay probabilities for the Λ_{e3} decay calculated from the graph in Fig. 2 for virtual K^--mesons undergoing K_{e3} decay in 2.

The values given in Table I for pseudovector K couplings are nearly of the necessary order of magnitude. If we further consider the K^0_{e3} decay which can cause exactly the same Λ decay mechanism with an amplitude which is about $\sqrt{2}$ times bigger than the one for K^-_{e3} decay and is superposed coherently on the later, we have to multiply the above values with a factor between 0.17 and 5.8. Then these values may give an appreciable part of the experimental value.

Hence we find the result that the empirical $\Lambda \to p + e^- + \bar{\nu}$ decay can partially be accounted for by indirect processes. There is no electron and neutrino asymmetry relative to Λ polarization direction because there are only strong couplings acting on the Λ. This is an important difference between our decay mechanism and that given by the direct $V-A$ coupling [9].

More detailed calculations will be published in *Acta Phys. Austr.*

* * *

I wish to thank Professor W. THIRRING for suggesting and discussing the problem and Dr. K. BAUMANN for his helpful remarks.

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[9] C. H. ALBRIGHT: *Phys. Rev.*, **115**, 750 (1959).

A New Method for Timing Scintillation Pulses.

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(ricevuto il 10 Febbraio 1960)

The « linear method » of working to get out a time information from a P. M. pulse has been introduced by one of the Authors ([1,2]) and extensively used ([3,4]). By this method a sinusoidal oscillation is linearly excited by the current pulse, and the pulse time position is evaluated by observing the phase of that oscillation at a time successive to the end of the pulse.

It is more convenient, especially using slow scintillators, as NaI(Tl), to take as reference time the first zero crossing of a sinusoidal oscillation whose period is shorter than the pulse decay time.

In this way the reference time is determined using only a small part of the photoelectrons related to a scintillation. Consequently statistical spread due to photoelectron emission is reduced. The following expression for ε_t^2 (machine time variance) is found when contributions due to ε_{ph}^2 (P.M. transit time variance) are neglected:

$$\varepsilon_t^2 = \frac{\tau T}{4R}(1+\varepsilon_A^2) \tag{1}$$

where:

ε_A^2 = P.M. gain variance;

R = photoelectron number related to a scintillation;

τ = scintillation decay time;

T = sinusoidal oscillation period.

This formula holds for T within the following limits:

$$T < \tau, \tag{2}$$

$$\frac{R}{\tau} T \gg 1 \tag{3}$$

When the lower limit (3) is not respected, ε_t^2 tends to the variance of the first photoelectron, given by (10) of ref. ([1]) for $Q=1$, that is:

$$\varepsilon_t^2 = \frac{\tau^2}{R^2}$$

In order to mantain the advantage of linear working, that is to discriminate

(*) Politecnico di Milano e Laboratori CISE.

(**) Laben.

([1]) S. Colombo, E. Gatti and G. Pignanelli: *Nuovo Cimento*, **5**, 1739 (1957).

([2]) E. Gatti and V. Svelto: *Nucl. Instr. and Methods*, **4**, 189 (1959).

([3]) C. Cottini and E. Gatti: *Nuovo Cimento*, **5**, 748 (1957).

([4]) F. T. Arecchi: *Energia Nucleare*. Vol. **6**, no. 11 (1959).

true pulses from the spurious thermoelectronic ones, it is convenient to use a period of oscillation such as to include a relatively high number of photoelectrons in the first semiperiod. This consideration gives a lower limit for T which is (3) again. For instance, using a NaI(Tl) crystal and a 40 MHz ringing frequency and taking into account standard photocathode efficiency, (3) limits to about 20 keV the lowest useful part of the spectrum. Expression (1) suggests to use the smallest value of T within the limit given by (3). Consequently the technical problem of an accurate zero crossing detection arises. Zero crossing detection methods were already described [4] but they become difficult increasing the frequency.

We propose the following system.

Anode and last dynode currents of the P.M. excite two damped resonant circuits whose frequencies are different from each other for a factor about two. The voltage outputs from the two ringing circuits drive the deflection plates of a C.R.T. causing a Lissajous figure.

We observe a narrow angular portion of these figures through a triangular slit whose vertex is on the rest point of the electronic spot.

Similar paths of different magnitude arise from different amplitudes of driving pulses, but every path is observed for an equal time interval and standard illumination pulses are so obtained (Fig. 1).

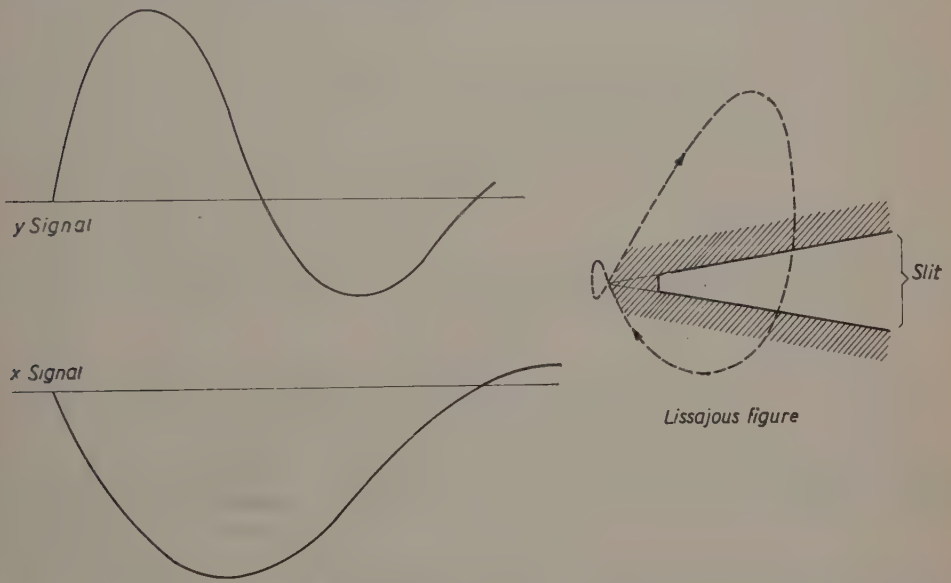

Fig. 1. – Waveforms on the CRT deflection plates, and Lissajous figure on the CRT screen.

A P.M. looking at the slit gives standard current pulses whose leading

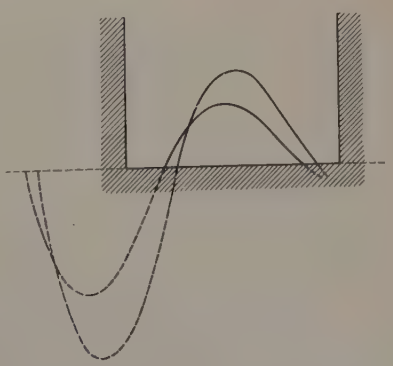

Fig. 2. – Waveforms on the CRT screen in the experimental arrangement.

edge is a ramp corresponding to the time of illumination, that is to the slit crossing time.

to the first zero crossing of the sinusoidal wave of shorter period.

We have made a preliminary check

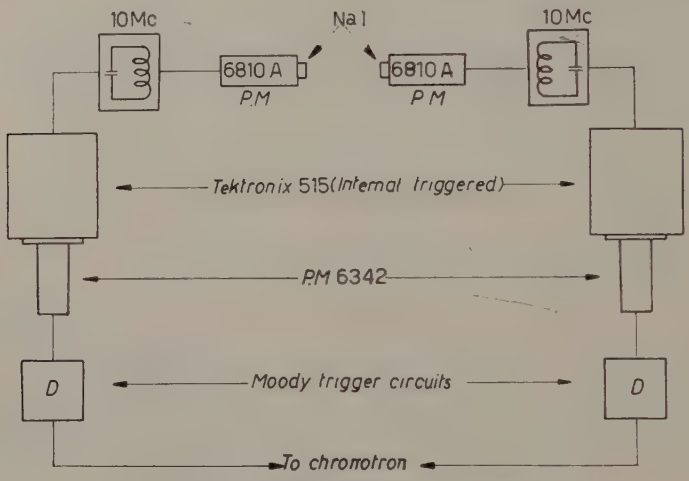

Fig. 3. – Experimental arrangement.

These pulses carry a time information whose variance is given in (1), when the P.M. looks at a zone corresponding of a timing arrangement similar to that proposed with an experimental set made with standard laboratory equipment.

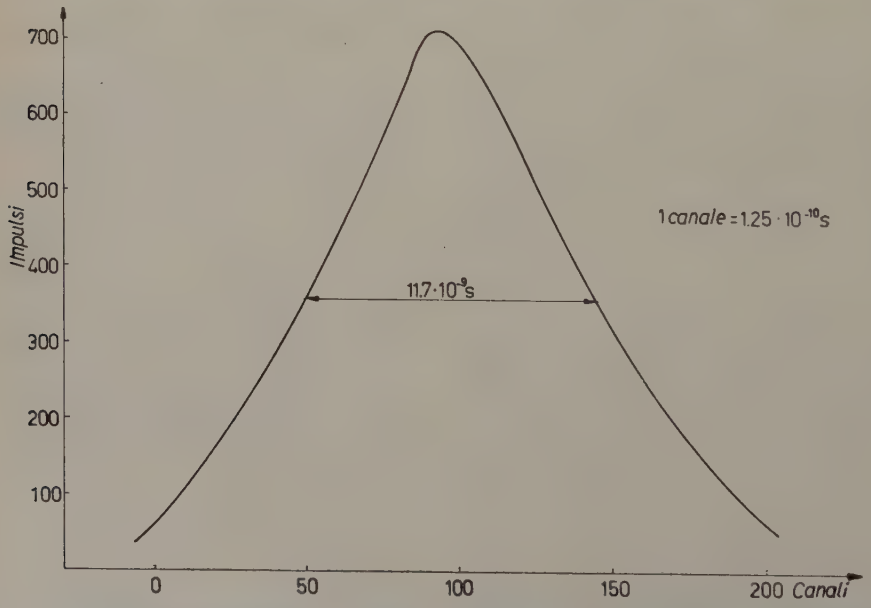

Fig. 4. – Prompt coincidence curve between ^{22}Na γ-rays - NaI(Tl) scintillator - Minimum energy accepted: 400 keV.

The current pulse from the P.M. anode is fed to a ringing circuit whose voltage output drives the vertical deflection plates of a C.R.T.

The time base starts with internal trigger. Amplitude fluctuations of the triggering signal will cause a jitter along the x-axis of the screen (Fig. 2). If the jitter is conveniently limited we can look through a convenient slit at a single half wave of each scintillation. Therefore we can get standard light pulses, which are converted by a photomultiplier in standard current pulses.

The time position of these standard pulses, detected by a Moody discriminator, is independent from scintillation amplitude, and its variance is given by (1).

We have obtained results in agreement with (1). The experimental assembly is given in Fig. 3 and in Fig. 4 an experimental coincidence curve for γ-rays of ^{22}Na registered with the vernier chronotron ([3]) is given. Our method is like the Morton and Matheson one ([5]). However it is different because the presence of two C.R.T. limiter-shapers enables to drive a time sorter. In addition the method of first zero crossing of a ringing circuit gives a variance for machine time, in the scale of Fig. 2 of ref. ([2]), $\frac{1}{2}(C/R)$ to be compared to C/R of the Morton and Matheson method and $\frac{1}{3}(C/R)$ if the centroid of the first C electrons could be detected.

* * *

We thank Mr. N. Carbone and Mr. G. Orlandi for their aid in assembling and testing the experimental set-

([5]) G. A. Morton and R. M. Matheson: *Nuclear Electronics* (International Atomic Energy Agency, 1959), p. 201.

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Analogy Between Lorentz and Foldy-Wouthuysen Transformation.

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We want to point out a peculiar analogy between both transformations, which comes out when one considers the velocity, instead as a parameter in the Lorentz transformation, as an appropriate operator.

Lorentz transformation is given by

$$L = \exp\left[\tfrac{1}{2}\,\mathrm{arctgh}\,\bar{\alpha}\bar{v}\right],$$

where the components of v are c-numbers.

Let us consider them instead as operators [1]

$$v = \frac{\beta p}{m},$$

which have the correct mean value.

When replaced in (1) we get:

$$L \to \exp\left[\frac{1}{2}\,\mathrm{arctgh}\,\frac{\alpha\beta p}{m}\right] = \exp\left[-\frac{1}{2}\,\frac{\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}\,\mathrm{arctg}\,\frac{p}{m}\right].$$

This is exactly the Foldy-transformation [2]. We also get the correct compact expression for the F-W transformation in presence of a magnetic field [3], by putting

$$v = \frac{\beta(\boldsymbol{p} - e\boldsymbol{A})}{m}.$$

We get then

$$F = \exp\left[\frac{1}{2}\,\mathrm{arctgh}\,\frac{\alpha\beta(p - eA)}{m}\right].$$

The analogy does not hold any more in presence of an electric field.

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[1] M. Bunge: *Nuovo Cimento*, **1**, 977 (1955); R. Feynman: *Caltec Notes* (1953).

[2] L. Foldy and S. A. Wouthuysen: *Phys. Rev.*, **78**, 1 (1950).

[3] K. M. Case: *Phys. Rev.*, **96**, 323 (1954).

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Anomalous Moment of the μ-Meson for Different Models of Breakdown of quantum Electrodynamics.

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(ricevuto il 2 Marzo 1960)

In a paper published some time ago, Berestetskij *et al.* [1], calculated the correction of the μ-meson anomalous magnetic moment due to a possible modification of the photon propagator in case of a break-down of quantum electrodynamics at small distances. Numerically, for the μ-meson, the correction is of the order of α ($\alpha = 1/137$) and therefore it adds directly to Schwinger's $\alpha/2\pi$ anomalous contribution (for the electron, because of its small mass, the deviation would be much smaller and unobservable).

The model that has been used consists in substituting to the photonic propagator $1/k^2$ the modified propagator

$$\frac{1}{k^2}\,\frac{\Lambda^2}{k^2+\Lambda^2} = \frac{1}{k^2} - \frac{1}{k^2+\Lambda^2}\,,$$

where Λ is the cut-off [2] ($\hbar = c = 1$), and it leads to the following results:

$$\frac{\delta\mu}{\mu} = \frac{\alpha}{2\pi}(1-\delta F)\,,$$

where $\mu = e/2m_\mu$ and

$$\delta F = 2\int_0^1 \frac{(1-x)\,x^2}{x^2-\eta x+\eta}\,\mathrm{d}x\,; \qquad \left(\eta = \frac{\Lambda^2}{m_\mu^2}\right),$$

$$\delta F \simeq \frac{2}{3}\,\frac{m_\mu^2}{\Lambda^2}\,. \qquad \text{for } \eta \gg 1\,.$$

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[1] V. B. Berestetskij, O. N. Krokhin and A. K. Khlebnikov: *Žurn. Èksp. Teor. Fiz.*, **30**, 788 (1956); *Journ. Exper. Theor. Phys.*, **3**, 76 (1956).

[2] See for ex. S. D. Drell: *Ann. Phys.*, **4**, 75 (1958).

In this note we evaluate what would be the correction to the muon magnetic moment from a possible modification either of the muon propagator or of the electromagnetic vertex. We first calculate the correction from a possible modification of the muon propagator. We substitute the modified propagator

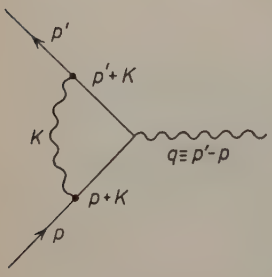

Fig. 1.

$$\frac{1}{(p+k)^2+m_\mu^2}\cdot\frac{\Lambda^2}{(p+k)^2+m_\mu^2+\Lambda^2},$$

to the usual fermion propagator in the Feynman graph of Fig. 1, and similarly for the propagator containing p', where p and p' are the 4-momenta of the ingoing and rsp. outgoing μ-meson, and the integration variable k, represents the photon 4-momentum. Putting in the final expression $p=p'$, we find the following correction to the anomalous moment

$$\frac{\delta\mu}{\mu}=\frac{\alpha}{2\pi}(1-\delta F'),$$

where

$$\delta F'=1-4\eta^2\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\frac{x(1-x)\,y^2 z}{[x+\eta(1-yz)]^3}=\frac{2}{3}\left\{\frac{1}{2}-\eta+\frac{1+\eta^3}{\eta}\ln(1+\eta)-\eta^2\ln\eta\right\},$$

and

$$\delta F'\simeq\frac{2}{3}\frac{1}{\eta}\left(\ln\eta+\frac{1}{3}\right).\qquad\text{for } \eta\gg 1.$$

We have assumed a possible vertex modification to depend for simplicity only from the virtual photon momentum k, according to the substitution

$$\gamma_\mu\to\frac{\Lambda^2}{k^2+\Lambda^2}\gamma_\mu.$$

Only two vertices must be modified. The vertex with external photon is not modified in the limit $q=0$ $(q=p'-p)$. We obtain

$$\frac{\delta\mu}{\mu}=\frac{\alpha}{2\pi}(1-\delta F''),$$

where

$$\delta F''=1-4\eta^2\int_0^1 dx\int_0^1 dy\,\frac{x^3y^2(1-x)(1-xy)}{[x^2y^2+\eta(1-x)]^3};$$

$$\delta F''=1-\frac{2}{1-4/\eta}\left\{3-\eta+\frac{\eta(\eta-4)}{2}\ln\eta-\frac{\eta^3-6\eta^2+6\eta+4}{2\eta\sqrt{1-4/\eta}}\ln\frac{1+\sqrt{1-4/\eta}}{1-\sqrt{1-4/\eta}}\right\},$$

for $\eta > 4$; and

$$\delta F'' = 1 - \frac{2}{1-4/\eta}\left\{3 - \eta + \frac{\eta(\eta-4)}{2}\ln\eta - \frac{\eta^3 - 6\eta^2 + 6\eta + 4}{\eta\sqrt{4/\eta - 1}}\operatorname{arctg}\sqrt{4/\eta - 1}\right\},$$

for $\eta < 4$. For $\eta \gg 1$

$$\delta F'' \simeq \frac{4}{3}\frac{1}{\eta}.$$

In the Table I we report the corrections δF corresponding for different values of the cut-off parameters η.

TABLE I.

η	δF	$\delta F'$	$\delta F''$
0	1	1	1
1	0.21	0.59	0.32
3	0.11	0.37	0.18
10	0.046	0.18	0.082
10^2	0.007	0.03	0.013
10^3	0.0007	0.005	0.001

* * *

Thanks are due to Prof. R. GATTO for having suggested the problem and for helpful discussion.

LIBRI RICEVUTI E RECENSIONI

Libri ricevuti.

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Recensioni.

J. W. Dungey – *Cosmic Electrodynamics*. Cambridge University Press, 1958, 32 s. 6 d., pp. ix-183.

La nuova scienza della magnetoidrodinamica ebbe un grande impulso una decade fa per opera di Alfvén con il suo volume *Cosmical Electrodynamics*. Oggi questa scienza ha raggiunto una notevole importanza e si è sviluppata una vasta letteratura sull'argomento. Con il nuovo volume, Dungey espone in maniera sobria e organica lo stato attuale (1956) della magnetoidrodinamica.

Il volume consiste di due parti: la prima tratta della magnetoidrodinamica vera e propria, comprendendo un capitolo sulla magnetostatica e uno sulla magnetodinamica dei fluidi conduttori. La seconda parte riguarda l'applicazione dei risultati ottenuti ai problemi della fisica solare, delle tempeste magnetiche e dell'elettrodinamica della ionosfera.

L'argomento è trattato in modo sin-

tetico ma completo. Sono discussi i principi fondamentali e sono segnalate le difficoltà e le limitazioni del metodo corpuscolare nel trattamento del problema. Grande importanza è data al problema della stabilità delle configurazioni di un fluido conduttore in campo magnetico. Questo problema, di interesse non solo per l'astrofisica, ma anche per il controllo delle reazioni termonucleari, è trattato soprattutto dal punto di vista delle applicazioni astrofisiche. Sono trattati brevemente i problemi delle onde d'urto idromagnetiche e della turbolenza idromagnetica.

Segue un'ottima descrizione e discussione dei fenomeni solari (macchie, brillamenti, protuberanze, regioni unipolari, ecc.) e delle loro relazioni con il campo magnetico solare.

Il capitolo sulle tempeste magnetiche si limita ad una buona esposizione della teoria di Chapman e Ferraro. Un capitolo è dedicato al problema dell'accelerazione cosmica di particelle cariche e in esso è inclusa la teoria dell'autore sui brillamenti solari.

È un volume utile e stimolante per chi si interessa al comportamento di fluidi conduttori in campo magnetico, e contiene una buona bibliografia (aggiornata al 1956).

S. SEGRE

Theorie und Molekulare Deutung Technologischer Eigenschaften von Hochpolymeren Werkstoffen. 4° volume della serie « Die Physik der Hochpolymeren », edito a cura di H. A. STUART, Springer-Verlag, Berlin - Göttingen - Heidelberg, 1956.

Questo libro costituisce una preziosa fonte di informazione per gli studiosi che si occupano di ricerche sui fenomeni molecolari ed in particolare sulle proprietà fisiche dei materiali.

Lo svolgimento dei dodici capitoli, curati da specialisti dei singoli rami, fornisce complessivamente una visione dello stadio attuale della conoscenza sulle teorie e sul significato molecolare delle proprietà fisiche di sostanze ad alto peso molecolare; in modo speciale vengono considerate le proprietà meccaniche ed elettriche dei polimeri allo stato solido, delle loro soluzioni e quelle di polimeri plastificati.

I tre primi capitoli trattano soprattutto delle teorie fenomenologiche sul comportamento viscoelastico lineare e non lineare e delle caratteristiche di resistenza a rottura. Tale comportamento viene completato con rappresentazione di modelli e con relazioni alla struttura molecolare.

Segue quindi un capitolo sui metodi tecnologici di prova sia dei materiali plastici e gommosi che delle fibre; di queste vengono discussi la reologia, i metodi di prova statici e dinamici. La critica dei metodi di prova delle materie plastiche fornisce ai tecnici un chiaro orientamento sulle reali misure fisiche rispetto ai metodi che forniscono solo dati empirici di ragguaglio.

Le proprietà meccaniche delle gomme vulcanizzate, specialmente in rapporto alla struttura molecolare ed alle teorie statistiche, le proprietà fotoelastiche, il rigonfiamento dei vulcanizzati, sono discusse esaurientemente nel quinto capitolo.

Una interessante esposizione sui rapporti tra struttura e proprietà meccaniche delle materie plastiche nei vari stati (solido, in soluzione, ecc.) e per diversi intervalli di temperatura (specialmente nella zona di transizione) forma l'oggetto di un successivo capitolo. In tale capitolo ed in quello seguente, dedicato alle fibre, gli autori hanno cercato di ordinare i dati sperimentali, fino ad oggi disponibili, in modo da fornire al lettore una rapida visione delle misure fisiche più significative e delle relazioni generali tra le costanti viscoelastiche con la struttura molecolare.

Nell'ottavo capitolo viene data una schematica rappresentazione delle proprietà elettriche (soprattutto quelle dielettriche) in funzione della struttura chimica e fisica dei polimeri, mentre nel seguente vengono discussi abbastanza diffusamente i fenomeni di plastificazione dei polimeri sia dal punto di vista applicativo che dal punto di vista strutturale.

Nell'ultimo capitolo H. A. Stuart e H. Mark riassumono le considerazioni sui rapporti tra struttura molecolare e proprietà macroscopiche di materiali costituiti da macromolecole filiformi. Il capitolo è diviso in tre sezioni; la prima tratta di una sistematica generale mentre la seconda e la terza sono dedicate rispettivamente alle fibre ed agli elastomeri.

Il libro è corredato da una completa bibliografia (aggiornata, in alcuni casi, fino al 1955) facilitando così al lettore il completamento degli argomenti trattati.

Accanto alle considerazioni teoretiche e fenomenologiche il libro porta nel suo assieme una visione critica dei dati sperimentali e dei metodi di misura; inoltre il significato delle relazioni tra la teoria, l'esperimento ed i fenomeni molecolari favorisce una trattazione fondamentale e completa del comportamento viscoelastico dei polimeri nei loro vari stati di esistenza. Come precisa l'autore nella sua prefazione, si è cercato di fornire allo studioso ed al tecnico in modo semplice e chiaro il punto sulla situazione attuale nel campo delle proprietà fisiche degli alti polimeri.

È indubbio che il libro diventerà per i ricercatori e per gli applicatori un testo di sempre maggiore importanza, per l'impostazione sicura dei problemi sulle materie plastiche, sulle fibre e sugli elastomeri ottenuti dai nuovi polimeri.

G. Crespi

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